A quantitative Hilbert's basis theorem and the constructive Krull dimension

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About me

Ryota Kuroki

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- My supervisor: Ryu Hasegawa
- Research interest: constructive algebra

What is constructive algebra?

Constructive algebra: Algebra without nonconstructive principles (e.g., excluded middle, Zorn's lemma, ...).

Constructive proofs have computational content. They can be regarded as programs for proof assistants.

Proof of $\exists n \in \mathbb{N}.\ \varphi(n) \rightsquigarrow$ Algorithm to compute n s.t. $\varphi(n)$

Noetherian rings (non-constructive)

Definition 1

A ring A is *Noetherian* if

$$\forall I_0 \leq I_1 \leq \cdots . \quad \exists n. \quad I_n = I_{n+1} = \cdots .$$

Example 1

1. All fields are Noetherian.
2. $\mathbb{Z}$ is Noetherian.
3. $\mathbb{Z}[X_0, X_1, \ldots]$ is not Noetherian.

$$0_{\mathbb{Z}} < \langle 12 \rangle < \langle 6 \rangle < \langle 3 \rangle < \langle 1 \rangle = \mathbb{Z}$$

$$0_{\mathbb{Z}[X_0, X_1, \ldots]} < \langle X_0 \rangle < \langle X_0, X_1 \rangle < \langle X_0, X_1, X_2 \rangle < \cdots$$

Noetherian rings (Richman–Seidenberg)

Problems:

- There are mysterious ideals like $\{x \in \mathbb{Z} : (x = 0) \vee \varphi\}$.
- $\langle 2 \rangle \leq \langle 2 \rangle \leq \cdots$
  (is it $\langle 2 \rangle$ forever? or will it be $\mathbb{Z}$ at somewhere?).

There are several constructive definition of Noetherianity (Buriola, Schuster, and Blechschmidt [2023]).
Definition by Richman [1974] and Seidenberg [1974]:

Definition 2

A ring A is *Noetherian* if

$$\forall I_0 \leq I_1 \leq \cdots \text{ (f.g.)} \quad \exists n. \quad I_n = I_{n+1}.$$

- If $I \leq \mathbb{Z}$ is f.g., we can compute $a \in \mathbb{Z}$ s.t. $I = \langle a \rangle$.
- We don't have to wait until I_n stabilizes.

Noetherian rings (Jacobsson–Löfwall)

Generalized inductive definition by Jacobsson and Löfwall [1991]:

Definition 3

An ideal $I \leq A$ is *blocked* if

$$\forall x \in A.\ (x \notin I) \to (I + \langle x \rangle \text{ is blocked}).$$

A ring A is *Noetherian* if $0 \leq A$ is blocked.

(I prefer $\forall x \in A.\ (x \in I) \vee (I + \langle x \rangle \text{ is blocked})$.)

$$\begin{tikzcd}
 & \langle 8 \rangle \arrow[r, dash] & \langle 4 \rangle \arrow[r, dash] & \langle 2 \rangle \arrow[dr, dash] & \\
0 \arrow[rr, dash] \arrow[drr, dash] & & \langle 6 \rangle \arrow[ur, dash] \arrow[r, dash] & \langle 3 \rangle \arrow[r, dash] & \mathbb{Z} \\
 & & \langle 10 \rangle \arrow[uur, dash] \arrow[r, dash] & \langle 5 \rangle \arrow[ur, dash] &
\end{tikzcd}$$

Noetherian rings (Coquand–Persson)

Generalized inductive definition by Coquand and Persson [1999]:

Definition 4

A list $[x_0, \dots, x_{n-1}] \in$ List A is *good* if $\exists k. x_k \in \langle x_0, \dots, x_{k-1} \rangle$.
A list $\sigma \in$ List A is *barred by good* if

$$(\sigma \text{ is good}) \vee (\forall x \in A.\ \sigma.x \text{ is barred by good}).$$

A ring is *Noetherian* if $[]$ is barred by good.

$$\begin{tikzcd}[arrows=dash]
 & {[2]} \arrow[r] \arrow[dr] & {\color{blue}[2,2]_{\text{good}}} \arrow[r] & {\color{blue}[2,2,1]_{\text{good}}} & \\
{[]_{\mathbb{Z}}} \arrow[ur] \arrow[dr] & & {[2,3]} \arrow[r] & {\color{blue}[2,3,1]_{\text{good}}} & \\
 & {[4]} \arrow[r] & {[4,2]} \arrow[r] & {[4,2,1]} \arrow[r] & {\color{blue}[4,2,1,1]_{\text{good}}}
\end{tikzcd}$$

α-Noetherian rings

Definition 5

A list $[x_0, \ldots, x_{n-1}]$ is (-1)-*good* (or simply *good*) if

$$(n \geq 1) \wedge x_{n-1} \in \langle x_0, \ldots, x_{n-2} \rangle.$$

A list $\sigma \in \mathsf{List}\, A$ is α-*good* ($\alpha \in \mathsf{Ord}$) if

$$\forall x \in A.\ \exists \beta \in [-1, \alpha).\ \sigma.x \text{ is } \beta\text{-good}.$$

A ring is α-*Noetherian* if $[]$ is α-good.

$$\begin{tikzcd}
 & {[2]_{\text{1-good}}} \arrow[r, dash] \arrow[dr, dash] & {\color{blue}[2,2]_{\text{good}}} \arrow[r, dash] & {[2,2,1]_{\text{0-good}}} & \\
{[]_{\omega\text{-good}}} \arrow[ur, dash] \arrow[dr, dash] & & {[2,3]_{\text{0-good}}} \arrow[r, dash] & {\color{blue}[2,3,1]_{\text{good}}} & \\
 & {[4]_{\text{2-good}}} \arrow[r, dash] & {[4,2]_{\text{1-good}}} \arrow[r, dash] & {[4,2,1]_{\text{0-good}}} \arrow[r, dash] & {\color{blue}[4,2,1,1]_{\text{good}}}
\end{tikzcd}$$

Classically, the notion of α-Noetherian ring is introduced by Gulliksen [1973] as the length of Noetherian modules.

Examples of α-Noetherian rings

Example 2

1. Discrete fields $(\forall x.\ (x = 0) \vee (x \in K^{\times}))$ are 1-Noetherian
2. $\mathbb{Z}$ is ω-Noetherian.

More generally, we can define α-Euclidean rings and prove that they are α-Noetherian. (Classically, the notion of α-Euclidean ring is essentially introduced by Motzkin [1949].)

Definition 6

1. $x \in A$ is called (-1)-*Euclidean* if $x = 0$.
2. $x \in A$ is called α-*Euclidean* if for every $y \in A$, there exist $\beta \in [-1, \alpha)$ and β-Euclidean element $z \in A$ s.t. $z - y \in \langle x \rangle$.
3. A ring A is called α-*Euclidean* if for every $x \in A$, there exists $\beta \in [-1, \alpha)$ s.t. x is β-Euclidean.

Hilbert's basis theorem (HBT)

Theorem 7 (Classical HBT)

In classical mathematics, if A is Noetherian, then so is $A[X]$.

Theorem 8 (Coquand–Persson HBT)

If A is Coquand–Persson Noetherian, then so is $A[X]$.

There are also Richman–Seidenberg HBT and Jacobsson–Löfwall HBT, but those are theorems about

$$\text{Noetherianity} + (\text{some conditions like coherence}),$$

which are classically equivalent to Noetherianity.

Quantitative Hilbert's basis theorem (QHBT)

Theorem 9 (Kuroki [2025])

If A is α-Noetherian, then $A[X]$ is $(\omega \otimes \alpha)$-Noetherian.

Classically, this is proved by Brookfield [2003].

Corollary 1

1. *If K is a discrete field, $K[X_0, \ldots, X_{n-1}]$ is ω^n-Noetherian.*
2. *$\mathbb{Z}[X_0, \ldots, X_{n-1}]$ is ω^{1+n}-Noetherian.*

Krull dimension (non-constructive)

Definition 10

We write Kdim $A < n$ if

$$\forall \mathfrak{p}_0 \leq \cdots \leq \mathfrak{p}_n.\ \exists k.\ \mathfrak{p}_k = \mathfrak{p}_{k+1}$$

Krull dimension (constructive)

Lombardi [2002] has found the following characterization:

Definition 11

We write Kdim $A < n$ if for every $x_0, \ldots, x_{n-1} \in A$, there exists $e_0, \ldots, e_{n-1} \geq 0$ such that

$$x_0^{e_0} \cdots x_{n-1}^{e_{n-1}} \in \langle x_0^{e_0+1},\ x_0^{e_0} x_1^{e_1+1}, \ldots,\ x_0^{e_0} \cdots x_{n-2}^{e_{n-2}} x_{n-1}^{e_{n-1}+1} \rangle.$$

As noted by [Lombardi, 2002, Proposition 5.2], this definition is closely related to the lexicographic order. For more results in this direction, see Kemper and Trung [2014], Kemper and Yengui [2020].

Example 3

1. Kdim $K < 1$ for a discrete field K.
2. Kdim $\mathbb{Z} < 2$.

α-Noetherianity and Krull dimension (1/2)

Theorem 12

Let $f : [0, \alpha) \to A$ be a function. If A is β-Noetherian for some $\beta < \alpha$, there exist $m \in \mathbb{N}$ and a strictly decreasing sequence $\alpha_0, \ldots, \alpha_{m-1} \in [0, \beta]$ s.t. $[f(\alpha_0), \ldots, f(\alpha_{m-1})]$ is good.

Proof.

Let $\alpha_0 := \beta$. Then $[f(\alpha_0)]$ is α_1-good for some $\alpha_1 \in [-1, \alpha_0)$.

1. If $\alpha_1 = -1$, then $[f(\alpha_0)]$ is good.
2. If $\alpha_1 \in [0, \alpha_0)$, then $[f(\alpha_0), f(\alpha_1)]$ is α_2-good for some $\alpha_2 \in [-1, \alpha_1)$...

□

α-Noetherianity and Krull dimension (2/2)

Theorem 13 (Classically proved by Gulliksen [1973])

If A is α-Noetherian for some $\alpha < \omega^n$, then Kdim $A < n$.

Proof.

Define $f : \omega^n \to A$ by $f(e_{n-1}, \ldots, e_1, e_0) := x_0^{e_0} \cdots x_{n-1}^{e_{n-1}}$. $\square$

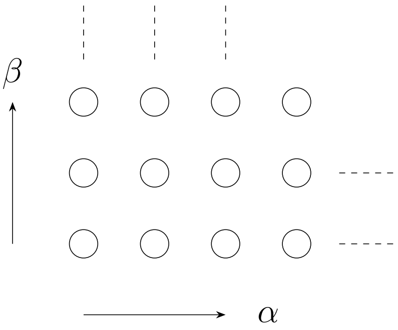
Corollary 2 (Lombardi [2002], Lombardi and Quitté [2015])

1. *If K is a discrete field,* Kdim $K[X_0, \ldots, X_{n-1}] < 1 + n$.
2. Kdim $\mathbb{Z}[X_0, \ldots, X_{n-1}] < 2 + n$.

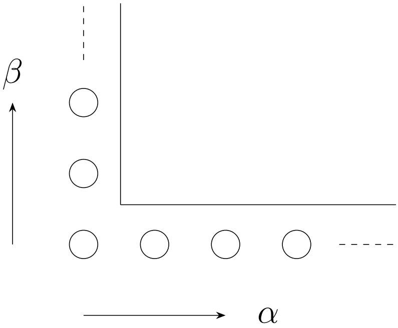
Transfinite Chomp (1/5)

To prove QHBT, we use a game called *(transfinite) chomp* (Huddleston and Shurman [2002]).

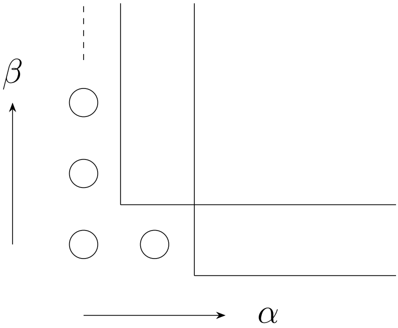
$(\alpha \times \beta)$-chomp:



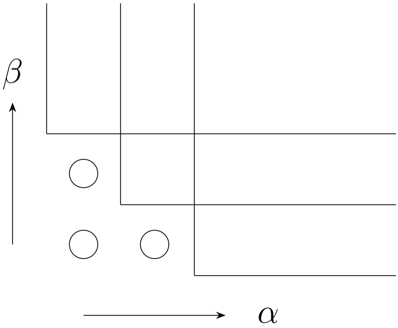
Transfinite Chomp (2/5)



Transfinite Chomp (3/5)



Transfinite Chomp (4/5)



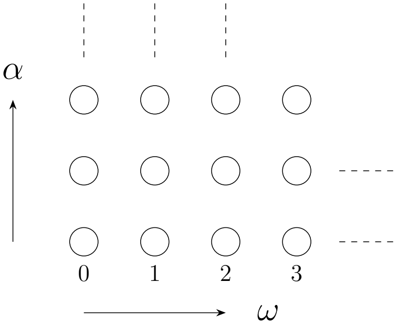
Transfinite Chomp (5/5)

The game ends in a finite number of steps. (Dickson's lemma)
Sketch of a proof (Huddleston and Shurman [2002]): We can assign an ordinal $\text{size}\, P$ to each position P of the game. Every time you remove circles, the size decreases. $\square$

The size of the initial position is $\alpha \otimes \beta$ (Hessenberg natural product).

Proof of QHBT (1/10)

Assume that A is α-Noetherian (i.e., $[]_A$ is α-good).

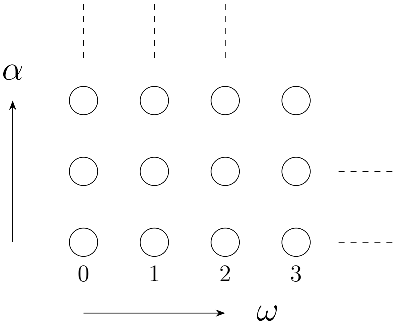


We prove that $[]$ is β_0-good, where

$$\beta_0 := (\text{size of the above position}) = \omega \otimes \alpha.$$

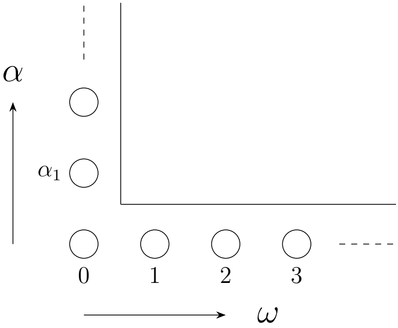
Suppose someone asks for $\beta_1 \in [-1, \beta_0)$ s.t. $\sigma_1 := [a_1 X + a_0]$ is β_1-good.

Proof of QHBT (2/10): find β_1 s.t. σ_1 is β_1-good



We received $\sigma_1 = [a_1 X + a_0]$.
We ask for $\alpha_1 \in [-1, \alpha)$ s.t. $[a_1]_A$ is α_1-good. Let's say $\alpha_1 = 1$.

Proof of QHBT (3/10): find β_1 s.t. σ_1 is β_1-good

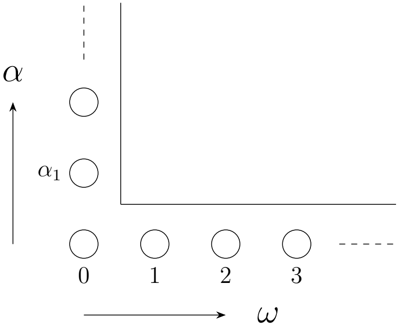


We remove the top-right area from the point $(1, 1)$.
Meaning: $\exists f_0 \in \langle \sigma_1 \rangle.\ \deg f_0 = 1 \wedge [\mathrm{lc}\, f_0]_A$ is 1-good.
We prove that $\sigma_1 = [a_1 X + a_0]$ is β_1-good, where

$$\beta_1 := (\text{size of the above position}).$$

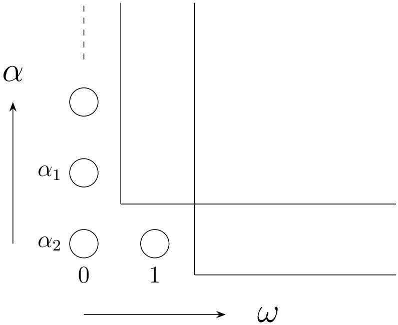
Suppose someone asks for $\beta_2 \in [-1, \beta_1)$ s.t.
$\sigma_2 := [a_1 X + a_0,\ b_2 X^2 + b_1 X + b_0]$ is β_2-good.

Proof of QHBT (4/10): find β_2 s.t. σ_2 is β_2-good



We received $\sigma_2 = [a_1X + a_0,\ b_2X^2 + b_1X + b_0]$.
We ask for $\alpha_2 \in [-1, \alpha_1)$ s.t. $[a_1, b_2]_A$ is α_2-good. Let's say $\alpha_2 = 0$.

Proof of QHBT (5/10): find β_2 s.t. σ_2 is β_2-good

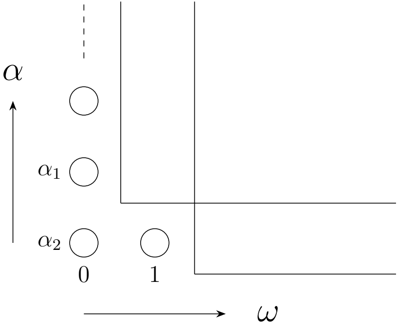


We remove the top-right area from the point $(2, 0)$.
Meaning: $\exists f_0, f_1 \in \langle \sigma_2 \rangle.\ \deg f_i = 2 \wedge [\mathrm{lc}\, f_0, \mathrm{lc}\, f_1]_A$ is 0-good.
We prove that $\sigma_2 = [a_1 X + a_0,\ b_2 X^2 + \cdots]$ is β_2-good, where

$$\beta_2 := (\text{size of the above position}).$$

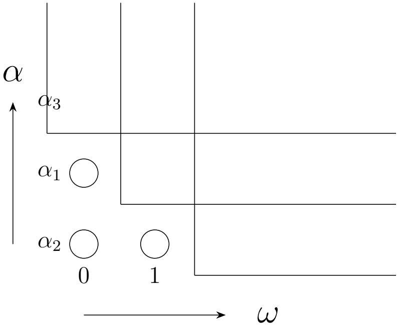
Suppose someone asks for $\beta_3 \in [-1, \beta_1)$ s.t.
$\sigma_3 := [a_1 X + a_0,\ b_2 X^2 + \cdots,\ c_0]$ is β_3-good.

Proof of QHBT (6/10): find β_3 s.t. σ_3 is β_3-good



We received $\sigma_3 = [a_1 X + a_0,\ b_2 X^2 + \cdots,\ c_0]$.
We ask for $\alpha_3 \in [-1, \alpha)$ s.t. $[c_0]_A$ is α_3-good. Let's say $\alpha_3 = 2$.

Proof of QHBT (7/10): find β_3 s.t. σ_3 is β_3-good

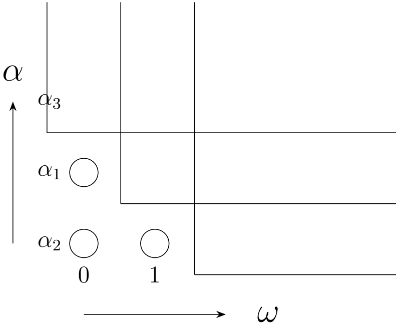


We remove the top-right area from the point $(2, 0)$.
Meaning: $\exists f_0 \in \langle \sigma_3 \rangle.\ \deg f_0 = 0 \wedge [\mathrm{lc}\, f_0]_A$ is 2-good.
We prove that $\sigma_3 = [a_1 X + a_0,\ b_2 X^2 + \cdots,\ c_0]$ is β_3-good, where

$$\beta_3 := (\text{size of the above position}).$$

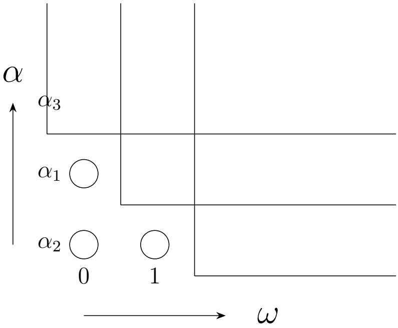
Suppose someone asks for $\beta_4 \in [-1, \beta_3)$ s.t.
$\sigma_4 := [a_1 X + a_0, \ldots, d_2 X^2 + d_1 X + d_0]$ is β_4-good.

Proof of QHBT (8/10): find β_4 s.t. σ_4 is β_4-good



We received $\sigma_4 = [a_1 X + a_0,\ b_2 X^2 + \cdots,\ c_0,\ d_2 X^2 + d_1 X + d_0]$.
We ask for $\alpha_4 \in [-1, \alpha_2)$ s.t. $[b_2, d_2]_A$ is α_4-good.
Then α_4 must be -1. Hence $d_2 \in \langle b_2 \rangle_A$. Hence
$\exists g \in A[X]. \deg g = 1 \wedge (g - (d_2 X^2 + d_1 X + d_0) \in \langle b_2 X^2 + \cdots \rangle)$.

Proof of QHBT (9/10): find β_4 s.t. σ_4 is β_4-good



Write g as $d_1'X + d_0'$.
We ask for $\alpha_4' \in [-1, \alpha_1)$ s.t. $[a_1, d_1']_A$ is α_4'-good...

Proof of QHBT (10/10)

Reduce the size of the position, reduce the degree of the polynomial at the end of the list, ...
By repeating this process, we can reduce the degree to -1. (When the size of the position reduces to 0, we have $1 \in \langle \sigma \rangle$.)

Hence $[]_{A[X]}$ is $(\omega \otimes \alpha)$-good. $\square$

Summary and future work

The notion of α-Noetherian ring works well with Krull dimension.

Future work: Constructive dimension theory of Noetherian rings

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