Reconciling

Brouwer's continuity principles

with Effectiveness

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"Synthetic mathematics, logic-affine computation and efficient proof systems."

Abstract

- What, Why, How

about the Minimalist Foundation (MF) for constructive maths

and its relation to Coquand-Huet's Calculus of Constructions (CC)

- consistency of both levels of MF $+$ Church Thesis of λ-functions (TCT)

  $+$ Brouwer's Continuity principles

- consistency of CC $+$ TCT $+$ Brouwer's Continuity principles

- Open problems

why developing Constructive Mathematics?

Bishop  wrote

the book "Foundations of constructive analysis"

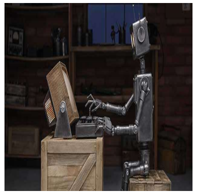
to show that *a large portion of functional analysis*

can be reproduced constructively

i.e.

with proofs having a computational contents

so that the existence of an object

can be computed by a machine 

and axiom of choice is valid

Essence of constructive mathematics

constructive mathematics

=

IMPLICIT computational mathematics

made EXPLICIT

in computable models

validating Church thesis CT

$+$ at least

number-theoretic Axiom of choice AC

for program-extraction from proofs

A principle of Effectiveness: Church Thesis

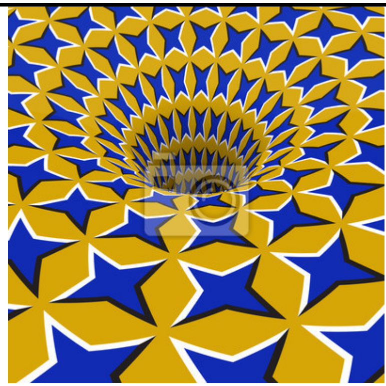
(**CT**)

$$\forall x \in \mathsf{N}at\ \exists!\, y \in \mathsf{N}at\ \ R(x,y)$$

$$\exists \mathbf{e} \in \mathsf{N}at \qquad (\forall x \in \mathsf{N}at\ \exists y \in \mathsf{N}at\ \ T(\mathbf{e},x,y)\ \&\ R(x,U(y)))$$

functional relations are all **computable**

Number-theoretic **Axiom of choice**



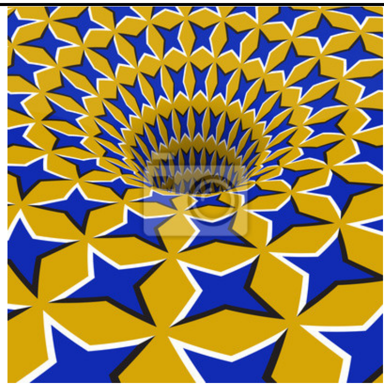
$(AC_{Nat,Nat})$

$$\forall x \in Nat\ \exists y \in Nat\ R(x,y) \quad \longrightarrow$$

$$\exists f \in Nat \to Nat\ \ \forall x \in Nat\ \ R(\,x\,,\ f(x)\,)$$

from any total relation we can extract a function

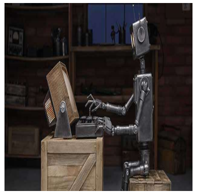
Axiom of choice



$$(AC) \quad \forall x \in A \ \exists y \in B \ R(x,y) \ \longrightarrow \ \exists f \in A \to B \ \ \forall x \in A \ \ R(\,x\,,\ f(x)\,)$$

from any total relation we can extract a function

Church Thesis is one discriminating property for constructivity



Consistency with Church Thesis

discriminates classical from constructive *arithmetics***:**

because

Heyting Arithmetics (=constructive Arithmetics)$+$ CT

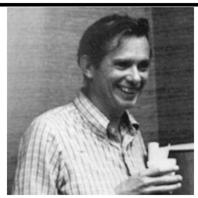
is consistent

being validated in Kleene realizability semantics

while classical Peano Arithmetics $+$CT $\vdash \perp$

i.e. inconsistent !

What best foundation for constructive mathematics ??



Since the 80s various foundations for **Bishop's constructive mathematics appeared including**

Martin-Löf's type theory

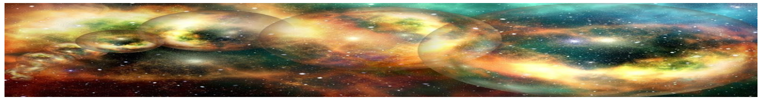
Aczel set theory

Homotopy type theory

. . .

with different behaviour w.r.t axiom of choice

Different behaviours of AC in constructive mathematics



in 1967, Bishop stated,

"A choice function exists in constructive mathematics,

because it is implied by the very meaning of existence"

... due to Brouwer-Heyting-Kolmogorov interpretation of INTUITIONISTIC logic

in 1975 Diaconescu proved that

in the internal logic of a topos AC $\Rightarrow$ *Excluded Middle*

in 1978 Goodman- Myhill proved that

in Constructive Set Theory AC $\Rightarrow$ *Excluded Middle*

What foundation for constructive mathematics ??

j.w.w. Giovanni Sambin

We wanted to take advantage of the plurality

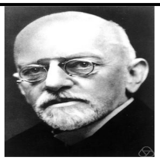
of **foundations** for **Bishop's constructive mathematics**

Plurality of foundations $\Rightarrow$ *need of a minimalist foundation*

	classical	constructive
	ONE standard	NO standard
impredicative	Zermelo-Fraenkel set theory	internal theory of topoi Coquand-Huet's Calculus of Constructions
predicative	Feferman's explicit maths	Aczel's CZF Martin-Löf's type theory **HoTT** and Voevodsky's Univalent Foundations Feferman's constructive expl. maths

the **Minimalist Foundation MF**

our foundational approach

as a **revised Hilbert program**: 

we need of a **trustable** foundation for **mathematics**

compatible with **most relevant foundations**

⇓

predicative à la **Weyl** 

constructive à la **Bishop** 

open-ended to further extensions according to **Martin-Löf** 

for **computed-aided formalization of its proofs** as advocated by **V. Voevodsky** 

Need of a two level Foundation for constructive mathematics

a **Constructive Foundation** should

bridge

extensional set-theoretic language

(commonly used for abstract math)

proof-irrelevant

to communicate with people

intensional language

for **computer-aided proof checking**

proof-relevant

to communicate with machines

$\downarrow$

realizability model for **program extraction from proofs**

Essence of Constructive mathematics and choice

Constructive Mathematics = maths which *admits* a COMPUTATIONAL interpretation

Constructive Mathematics is a

bridge

abstract maths

no need of choice function

COMPUTATIONAL maths

yes, need of choice function

for extraction of programs

from proofs

Our Proposal (j.w.w. G. Sambin) of constructive foundation

better to found mathematics in a TWO-LEVEL FOUNDATION

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extensional level (used by mathematicians to developed their proofs)

$\Downarrow$ interpreted restoring intensional information

(via a quotient completion)

intensional level (language of TYPE THEORY)

Our TWO-LEVEL Minimalist Foundation

from [Maietti'09] in agreement with [M. Sambin2005]

its intensional level mTT

(Minimalist Type Theory)

= a PREDICATIVE VERSION of Coquand-Huet's Calculus of Constructions CC

(fragment of Rocq)

= first order Martin-Löf's intensional type theory + primitive propositions

+ one UNIVERSE of small propositions

its extensional level emTT

(extensional Minimalist Type Theory)

is a PREDICATIVE LOCAL set theory

(NO choice principles)

How to gain CC from the intensional level of MF

Theorem:

Coquand-Huet's Calculus of Constructions CC

(with *list types, finite disjoint sums***)**

which is impredicative

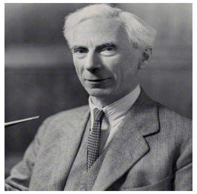
is equivalent to $\Updownarrow$

the intensional level mTT of MF

$+$ **the following resizing rules**

$$\frac{\phi \text{ proposition}}{\phi \text{ small proposition}} \qquad \frac{A \text{ col}}{A \text{ set}}$$

Russell’s notion of predicative definitions



“Whatever involves an apparent variable

must *not be among the possible values* **of that variable.”**

What is the impredicative analogoue of the extensional level of MF

Theorem:

The ***internal generic theory*** **of quasi-toposes**

which is impredicative

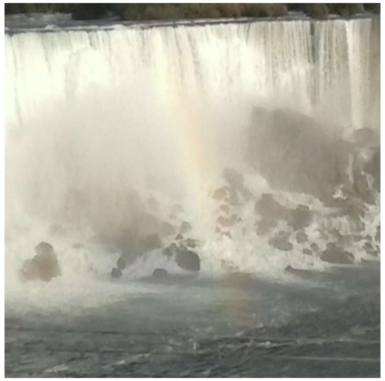
is equivalent to $\Updownarrow$

the extensional level emTT of MF

$+$ **the following resizing rules**

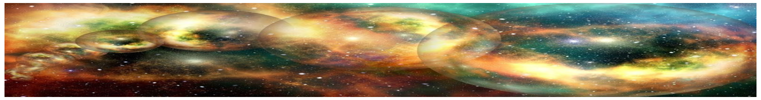
$$\frac{\phi \text{ proposition}}{\phi \text{ small proposition}} \qquad \frac{A \text{ col}}{A \text{ set}}$$

reading AC in the two-level Minimalist Foundation



EXTENSIONAL level of MF:	AC= Zermelo axiom of choice
	$\Downarrow$ gets interpreted into
INTENSIONAL level of MF :	Martin-Löf's extensional axiom of choice NOT constructively acceptable $\Rightarrow$ classical logic

Extensional Axiom of Choice



$$(AC_{ext}) \qquad \forall x \in A \ \exists y \in B \ R(x,y) \quad \longrightarrow$$
$$\exists f \in A \rightarrow B \quad (\ \mathbf{Ext}(\mathbf{f})^{\simeq_{\mathbf{B}}}_{\simeq_{\mathbf{A}}} \ \ \& \ \ \forall x \in A \ \ R(\,x\,,\ f(x)\,)\,)$$

from any total relation we can extract a function preserving arbitrary $\simeq_A$ and $\simeq_B$

provided that $R(x,y)$ preserves $\simeq_A$ and $\simeq_B$

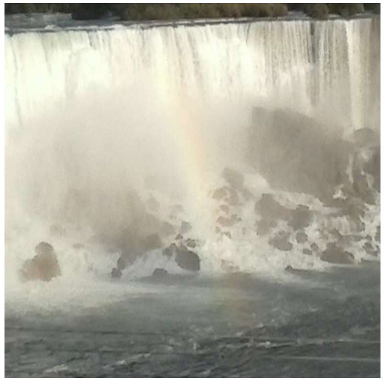
i.e.

$$\mathbf{Ext}(\mathbf{f})^{\simeq_{\mathbf{B}}}_{\simeq_{\mathbf{A}}} \ \equiv \ \forall x_1 \in A \, \forall x_2 \in A \, (\, x \simeq_A y \ \rightarrow \ f(x_1) \simeq_B f(x_2)\,)$$

NO constructive

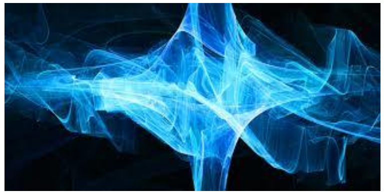
$\Rightarrow$ classical logic

What corresponds to AC of the intensional level?



Extensional level of MF	**Axiom of unique choice**
	$\Updownarrow$
Intensional level of MF:	**Axiom of choice AC** **provable in Martin-Loef's type theory** **as *advocated* by Bishop**

two notions of function in the Minimalist Foundation



a *primitive notion* of **type-theoretic function**

$$f(x) \in B \; [x \in A]$$

(closed under "exponentiation")

$\neq$ **(syntactically)**

notion of functional relation

$$\forall x \in A \; \exists! y \in B \; R(x, y)$$

(NOT closed under "exponentiation")

Axiom of unique choice

$$\forall x \in A\, \exists! y \in B\, R(x,y) \quad \longrightarrow \quad \exists f \in A \to B\, \forall x \in A\, R(x, f(x))$$

turns a functional relation into a type-theoretic function.

$\Rightarrow$ **identifies the two distinct notions...**

A peculiarity of the intensional level of MF and MLTT: consistency with $\mathbf{CT} + \mathbf{AC}$

consistency of the intensional level of MF + Church Thesis CT $+$ Axiom of Choice AC

by "an extension of Kleene realizability"

with Feferman's theory of non-iterative fixtpoints

in H. Ishihara, M.E.M., S. Maschio, T. Streicher, AML, 2018

extended to a constructive consistency of

intensional level of MF $+$inductive/co-inductive definitions $+$ $\mathbf{CT}$ $+$ AC

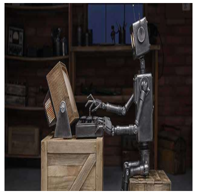
in CZF+REA shown in

M.E.M. , S. Maschio and M. Rathjen , LMCS, 2021

M.E.M. , S. Maschio and M. Rathjen, LMCS 2022

M.E.M. , P. Sabelli, MFPS'23 $+$ P. Sabelli, PhD thesis, 2024

A peculiarity of MLTT



also intensional Martin-Löf's type theory MLTT is consistent with Church Thesis CT + Axiom of Choice AC
but Homotopy Type Theory is NOT since $\mathbf{HoTT} + \mathbf{CT} +$ propositional AC $\vdash \perp$

Church Thesis CT + Axiom of Choice AC contradicts extensionality

Heyting arithmetics with finite types $+$ **AC+CT+ extfun** $\vdash \perp$

where

$$\text{extfun}\ \frac{f(x) =_B g(x)\ \ true\ \ [x \in A]}{\lambda x.f(x) =_{A \to B} \lambda x.g(x)\ \ true}$$

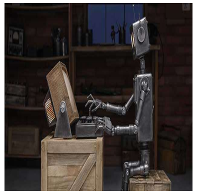
extensionality of functions

A realizability semantics for the extensional level of MF + Church Thesis + AC$_{Nat,Nat}$

extensional level of MF+ Church Thesis +AC$_{nat,nat}$

$\Downarrow$ (interpreted)

predicative Hyland's Effective Topos



in [Maietti-Maschio'21] "A predicative variant of Hyland's Effective Topos" JSL 2021

built out of a categorical structure

(= the elementary quotient completion from [M.-Rosolini13])

using the realizability semantics for the intensional level of MF

Open issue

What model for

Coquand-Huet's Calculus of Constructions + CT + AC?

or even

Is the consistency **+ CT + AC**

discriminating **predicative from impredicative theories?**

Brouwer's intuitionism contradicts Church thesis

Brouwer's continuity principles + Church thesis $\vdash \perp$

($\Rightarrow$ incompatible with Russian constructivism including CT)

because

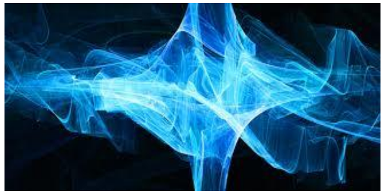
each level of MF $+$ Fan theorem $+$CT is inconsistent

where

Fan theorem = *spatiality* of Cantor space

choice sequences=functional relations

two notions of function in the Minimalist Foundation



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$$f(x) \in B \; [x \in A]$$

(closed under "exponentiation")

$\neq$ **(syntactically)**

notion of functional relation

$$\forall x \in A \; \exists! y \in B \; R(x, y)$$

(NOT closed under "exponentiation")

Type-theoretic Church thesis

(**TCT**)

$$\forall f \in \mathsf{N}at{\rightarrow}\mathsf{N}at \qquad \exists e \in \mathsf{N}at$$

$$(\,\forall x \in \mathsf{N}at \;\; \exists y \in \mathsf{N}at \;\; T(e,x,y) \;\&\; U(y) =_{\mathsf{N}at} f(x)\,)$$

type-theoretic functions ($\subset_{\neq}$ **functional relations**)

are all **computable**

Peculiariy of MF: reconciling Russian constructivism with Brouwer intuitionism

Theorem:

Both levels of MF are consistent with $+$Theoretical Church Thesis$(\mathbf{TCT})$

+ Brouwer's continuity principles

Bar Induction (BI) =spatiality of Baire locale
Local Continuity Principle (LCP)= continuity of functions from Baire space to Nat

where $\boxed{\text{Brouwer's choice sequences= functional relations}}$

Proof: a model made of Hyland's assemblies formalized in

Aczel's CZF+REA + BI +LCP

(= a predicative and constructive quasi-topos of assemblies!!!)

Decomposition of Church Thesis

CT
$\Updownarrow$
TCT $+$ AC$!_{Nat,Nat}$

Axiom of unique choice

$$\forall x \in A\, \exists ! y \in B\, R(x,y) \longrightarrow \exists f \in A \to B\, \forall x \in A\, R(x, f(x))$$

turns a functional relation into a type-theoretic function.

$\Rightarrow$ identifies the two distinct notions...

valid in Homotopy Type Theory, Martin-Löf's type theory and Aczel's CZF

but NOT in the Minimalist Foundation

Foundations NOT reconciling Russian constructivism with Brouwer's intuitionism

Homotopy type theory and Martin-Löf's type theory

do NOT satisfy such a consistency property

because

HoTT $+$ $\mathbf{BI} + \mathbf{LCP}$ $+$ TCT $\vdash \perp$

Martin-Löf's type theory $+ \mathbf{BI} + \mathbf{LCP} +$ TCT $\vdash \perp$

due to Kleene's proof that

since they validate AC!

Peculiariy of CC: reconciling Russian constructivism with Brouwer intuitionism

Theorem:

CC is consistent with $+$Theoretical Church Thesis$(\mathbf{TCT})$

+ Brouwer's continuity principles:

Bar Induction $($BI$)$ =spatiality of Baire locale

Local Continuity Principle $($LCP$)$= continuity of functions from Baire space to Nat

where Brouwer's choice sequences= functional relations

Proof. a model made of Hyland's assemblies formalized in

IZF + BI +LCP

(= an intuitionistic quasi-topos of assemblies!!!)

the two levels of MF are equiconsistent

from j.w.w. Pietro Sabelli in Apal 2024

emTT + propositional univalence $\Updownarrow$ $\Downarrow$ **in [M09]** $\Uparrow$ **as in [M09]**

$(\mathsf{Props})^{int}$ **as** $\mathcal{P}(1)$

mTT

propositional univalence

$$\frac{\phi \ \textbf{prop} \qquad \psi \ \textbf{prop} \qquad \phi \leftrightarrow \psi \ \textbf{true}}{\phi = \psi \ prop}$$

CC is equiconsistent to the internal language of quasi-toposes

from j.w.w. Pietro Sabelli in Apal 2024

int. lang. **of quasi-toposes + propositional univalence**

$\Updownarrow$ $\Downarrow$ **as in [M09]** $\Uparrow$ **as in [M09]**

but $(\mathsf{Props})^{int}$ **as** $\mathcal{P}(1)$

CC

propositional univalence

$$\frac{\phi \ \textbf{prop} \qquad \psi \ \textbf{prop} \qquad \phi \leftrightarrow \psi \ \textbf{true}}{\phi = \psi \ prop}$$

Open issues

• **Computer-formalization of the two-level MF**
• **Consistency of Coquand-Huet's Calculus of Constructions + CT + AC**   (*Is this a property discriminating predicative from impredicative theories*?)
• **Consistency of the intensional level of MF + AC + Brouwer's continuity principles**   **(with NO ξ-rule)**