Equivalences between categories with dependent arrows and comprehension categories

Luis Gambarte

LMU München

CIRM 3377
Synthetic mathematics, logic-affine computation and efficient proof systems
12.09.2025

Overview

$$
\begin{tikzcd}
 & (\mathrm{dep}, \Sigma)\mathrm{C} \arrow[rr] & & (2\text{-}\mathrm{dep}, \Sigma)\mathrm{C} \\
\mathrm{depC} \arrow[ur] \arrow[rr] & & 2\text{-}\mathrm{depC} \arrow[ur] & \\
 & (\mathrm{fam}, \Sigma)\mathrm{C} \arrow[uu] \arrow[rr] & & (2\text{-}\mathrm{fam}, \Sigma)\mathrm{C} \arrow[uu] \\
\mathrm{famC} \arrow[uu] \arrow[ur] \arrow[rr] & & 2\text{-}\mathrm{famC} \arrow[uu] \arrow[ur] &
\end{tikzcd}
$$

Overview

$$
\begin{tikzcd}
 & \mathrm{HComprC}^{\mathrm{str}}_{\mathrm{disc}} \arrow[rr] & & ? \\
\mathrm{dFibb} \arrow[rr] \arrow[ur] & & \mathrm{aFibd} \arrow[ur] & \\
 & \mathrm{ComprC}_{\mathrm{disc}} \arrow[rr] \arrow[uu] & & \mathrm{ComprC}^{\mathrm{str}}_{\mathrm{spl}} \arrow[uu] \\
\mathrm{dFib} \arrow[rr] \arrow[uu] \arrow[ur] & & \mathrm{Fib} \arrow[uu] \arrow[ur] &
\end{tikzcd}
$$

Motivation

In Type Theory:

- families of small types $F : A \to \mathcal{U}$

Motivation

In Type Theory:

- families of small types $F : A \to \mathcal{U}$
- precomposition with terms of arrow type $f : B \to A$

$$B \xrightarrow{f} A \xrightarrow{F} \mathcal{U}$$

Motivation

In Type Theory:

- families of small types $F : A \to \mathcal{U}$
- precomposition with terms of arrow type $f : B \to A$

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[rr, bend right, "F \circ f"'] & A \arrow[r, "F"] & \mathcal{U} \end{tikzcd}$$

Motivation

In Type Theory:

- families of small types $F : A \to \mathcal{U}$
- precomposition with terms of arrow type $f : B \to A$

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[rr, bend right, "F \circ f"'] & A \arrow[r, "F"] & \mathcal{U} \end{tikzcd}$$

-

$$\begin{tikzcd} B \arrow[r, "f"] & A \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{U} \end{tikzcd}$$

Motivation

In Type Theory:

- families of small types $F : A \to \mathcal{U}$
- precomposition with terms of arrow type $f : B \to A$

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[rr, bend right, "F \circ f"'] & A \arrow[r, "F"] & \mathcal{U} \end{tikzcd}$$

-

$$\begin{tikzcd} B \arrow[r, bend left, "F \circ f"] \arrow[r, bend right, "G \circ f"'] \arrow[r, phantom, "\Downarrow \eta \circ f"] & \mathcal{U} \end{tikzcd}$$

Motivation

- Dependent function $\Phi : \prod_{x:A} F(x)$

$$\begin{tikzcd} B \arrow[r, "f"] & A \arrow[r, "F"'] \arrow[r, bend left, Rightarrow, "\Phi"] & \mathcal{U} \end{tikzcd}$$

Motivation

- Dependent function $\Phi : \prod_{x:A} F(x)$

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[rr, bend right, "F\circ f"'] & A \arrow[r, "F"] \arrow[r, Rightarrow, bend left, "\Phi"] & \mathcal{U} \end{tikzcd}$$

Motivation

- Dependent function $\Phi : \prod_{x:A} F(x)$

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[rr, bend right, "F\circ f"'] \arrow[rr, phantom, bend right=60, "\Phi\circ f \Rightarrow"] & A \arrow[r, "F"] \arrow[r, phantom, bend left=40, "\Phi \Rightarrow"] & \mathcal{U} \end{tikzcd}$$

$\Phi \circ f : \prod_{y:B} F\big(f(y)\big)$

Motivation

- Dependent function $\Phi : \prod_{x:A} F(x)$

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[rr, bend right, "\Phi\circ f \Rightarrow F\circ f"'] & A \arrow[r, "F"] \arrow[r, bend left, phantom, "\Phi"] & \mathcal{U} \end{tikzcd}$$

  $\Phi \circ f : \prod_{y:B} F\big(f(y)\big)$

-

$$\begin{tikzcd} B \arrow[r, "f"] & A \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{U} \end{tikzcd}$$

Motivation

- Dependent function $\Phi : \prod_{x:A} F(x)$

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[rr, bend right, "\Phi\circ f \Rightarrow F\circ f"'] & A \arrow[r, "F"'] \arrow[r, phantom, bend left, "\Phi \Rightarrow"] & \mathcal{U}
\end{tikzcd}$$

$\Phi \circ f : \prod_{y:B} F\big(f(y)\big)$

- $$\begin{tikzcd}
B \arrow[r, bend left, "F\circ f"] \arrow[r, bend right, "G\circ f"'] \arrow[r, phantom, "\Downarrow \eta\circ f"] & \mathcal{U}
\end{tikzcd}$$

Motivation

- Dependent function $\Phi : \prod_{x:A} F(x)$

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[rr, bend right, "F\circ f"] \arrow[rr, phantom, bend right=60, "\Phi\circ f"'] & A \arrow[r, "F"] \arrow[r, phantom, bend left=60, "\Phi"] & \mathcal{U} \end{tikzcd}$$

$\Phi \circ f : \prod_{y:B} F\big(f(y)\big)$

- $$\begin{tikzcd} B \arrow[r, bend left, "F\circ f"] \arrow[r, bend right, "G\circ f"'] \arrow[r, phantom, bend left=70, "\Phi\circ f"] \arrow[r, phantom, "\Downarrow \eta\circ f"] & \mathcal{U} \end{tikzcd}$$

Motivation

- Dependent function $\Phi : \prod_{x:A} F(x)$

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[rr, bend right, "F\circ f"'] \arrow[rr, phantom, bend right=60, "\Phi\circ f \Rightarrow F\circ f"] & A \arrow[r, "F"] \arrow[r, phantom, bend left=50, "\Phi \Rightarrow F"] & \mathcal{U} \end{tikzcd}$$

$\Phi \circ f : \prod_{y:B} F\big(f(y)\big)$

-

$$\begin{tikzcd} B \arrow[r, bend left, "F\circ f"] \arrow[r, bend right, "G\circ f"'] \arrow[r, phantom, "\Downarrow \eta\circ f"] \arrow[r, phantom, bend left=70, "\Phi\circ f \Rightarrow F\circ f"] \arrow[r, phantom, bend right=70, "(\eta\circ\Phi)\circ f \Rightarrow G\circ f"'] & \mathcal{U} \end{tikzcd}$$

(2-)fam-arrow structures

Def. (fam-categories) $\mathscr{C}$ category
+ $\mathrm{fHom}(c)$ for $c \in \mathscr{C}$

$$c \xrightarrow{\lambda}$$

(2-)fam-arrow structures

Def. (fam-categories) $\mathscr{C}$ category
+ $\mathrm{fHom}(c)$ for $c \in \mathscr{C}$

$$c \xrightarrow{\lambda}$$

+ $\circ\colon (c) \times \mathrm{Hom}(c', c) \to \mathrm{fHom}(c')$

$$c' \xrightarrow[f]{} c \xrightarrow[\lambda]{}$$

(2-)fam-arrow structures

Def. (fam-categories) $\mathscr{C}$ category
+ $\mathrm{fHom}(c)$ for $c \in \mathscr{C}$

$$\begin{tikzcd} c \arrow[r, "\lambda"] & {} \end{tikzcd}$$

+ $\circ\colon (c) \times \mathrm{Hom}(c', c) \to \mathrm{fHom}(c')$

$$\begin{tikzcd} c' \arrow[r, "f"'] \arrow[rr, bend left, "\lambda \circ f"] & c \arrow[r, "\lambda"'] & {} \end{tikzcd}$$

1. $\lambda \circ 1_c = \lambda$
2. $\lambda \circ (g \circ f) = (\lambda \circ g) \circ f$

(2-)fam-arrow structures

Def. (fam-categories) $\mathscr{C}$ category
+ $\mathrm{fHom}(c)$ for $c \in \mathscr{C}$

$$\begin{tikzcd} c \arrow[r, "\lambda"] & {} \end{tikzcd}$$

+ $\circ\colon\ (c) \times \mathrm{Hom}(c', c) \to \mathrm{fHom}(c')$

$$\begin{tikzcd} c' \arrow[r, "f"'] \arrow[rr, bend left, "\lambda\circ f"] & c \arrow[r, "\lambda"'] & {} \end{tikzcd}$$

1. $\lambda \circ 1_c = \lambda$
2. $\lambda \circ (g \circ f) = (\lambda \circ g) \circ f$

Def. (discrete fibrations) $p\colon \mathscr{E} \to \mathscr{B}$

$$\begin{tikzcd} & e \arrow[d, mapsto, "p"] \\ b' \arrow[r, "f"'] & b \end{tikzcd}$$

(2-)fam-arrow structures

Def. (fam-categories) $\mathscr{C}$ category

$+$ $\mathrm{fHom}(c)$ for $c \in \mathscr{C}$

$$c \xrightarrow{\lambda}$$

$+ \circ \colon (c) \times \mathrm{Hom}(c', c) \to \mathrm{fHom}(c')$

$$\begin{tikzcd}
c' \arrow[r, "f"'] \arrow[rr, bend left, "\lambda\circ f"] & c \arrow[r, "\lambda"'] & {}
\end{tikzcd}$$

1. $\lambda \circ 1_c = \lambda$
2. $\lambda \circ (g \circ f) = (\lambda \circ g) \circ f$

Def. (discrete fibrations) $p \colon \mathscr{E} \to \mathscr{B}$

$$\begin{tikzcd}
f^*(e) \arrow[r, dotted, "\exists!\overline{f}(e)"] \arrow[d, mapsto, "p"'] & e \arrow[d, mapsto, "p"] \\
b' \arrow[r, "f"'] & b
\end{tikzcd}$$

(2-)fam-arrow structures

Def. (2-fam-categories) $\mathscr{C}$ fam-category

(2-)fam-arrow structures

Def. (2-fam-categories) $\mathscr{C}$ fam-category
+ $\mathrm{fHom}(c)$ for $c \in \mathscr{C}$

$$\begin{tikzcd} c \arrow[r, bend left, "\lambda"{name=U}] \arrow[r, bend right, "\mu"'{name=D}] & {} \arrow[from=U, to=D, Rightarrow, "\eta"] \end{tikzcd}$$

(2-)fam-arrow structures

Def. (2-fam-categories) $\mathscr{C}$ fam-category

+ $\mathrm{fHom}(c) \leftarrow$**category**

$$\begin{tikzcd} c \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\mu"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \end{tikzcd}$$

$$+ \circ\colon\ \mathrm{fHom}(c) \times \mathrm{Hom}(c', c) \to \mathrm{fHom}(c')$$

$$\begin{tikzcd} c' \arrow[r, "f"] & c \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\mu"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \end{tikzcd}$$

(2-)fam-arrow structures

Def. (2-fam-categories) $\mathscr{C}$ fam-category
+ $\mathrm{fHom}(c)$ ←**category**

$$\begin{tikzcd} c \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\mu"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \end{tikzcd}$$

$$+\ \circ\colon\ \mathrm{fHom}(c) \times \mathrm{Hom}(c', c) \to \mathrm{fHom}(c')$$

$$\begin{tikzcd} c' \arrow[r, bend left, "\lambda\circ f"] \arrow[r, bend right, "\mu\circ f"'] \arrow[r, phantom, "\Downarrow\eta\circ f"] & {} \end{tikzcd}$$

(2-)fam-arrow structures

Def. (2-fam-categories) $\mathscr{C}$ fam-category
+ $\mathrm{fHom}(c)$ ←**category**

$$\begin{tikzcd} c \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\mu"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \end{tikzcd}$$

$$+ \circ\colon\ \mathrm{fHom}(c) \times \mathrm{Hom}(c', c) \to \mathrm{fHom}(c')$$

$$\begin{tikzcd} c' \arrow[r, "f"] & c \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\mu"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \end{tikzcd}$$

1. $\eta \circ 1_c = \eta$

(2-)fam-arrow structures

Def. (2-fam-categories) $\mathscr{C}$ fam-category
+ $\mathrm{fHom}(c) \leftarrow$ **category**

$$\begin{tikzcd} c \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\mu"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \end{tikzcd}$$

+ $\circ\colon \ \mathrm{fHom}(c) \times \mathrm{Hom}(c', c) \to \mathrm{fHom}(c')$

$$\begin{tikzcd} c' \arrow[r, "f"] & c \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\mu"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \end{tikzcd}$$

1. $\eta \circ 1_c = \eta$
2. $\eta \circ (f \circ g) = (\lambda \circ f) \circ g$

Def. (split fibrations)
$p\colon \mathscr{E} \to \mathscr{B}$

$$\begin{tikzcd} & e \arrow[d, mapsto, "p"] \\ b' \arrow[r, "f"'] & b \end{tikzcd}$$

(2-)fam-arrow structures

Def. (2-fam-categories) $\mathscr{C}$ fam-category
+ $\mathrm{fHom}(c)$ ←**category**

$$\begin{tikzcd} c \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\mu"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \end{tikzcd}$$

$$+ \circ\colon\ \mathrm{fHom}(c) \times \mathrm{Hom}(c', c) \to \mathrm{fHom}(c')$$

$$\begin{tikzcd} c' \arrow[r, "f"] & c \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\mu"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \end{tikzcd}$$

1. $\eta \circ 1_c = \eta$
2. $\eta \circ (f \circ g) = (\lambda \circ f) \circ g$

Def. (split fibrations)
$p\colon \mathscr{E} \to \mathscr{B}$

$$\begin{tikzcd} f^*(e) \arrow[r, dotted, "\exists \overline{f}(e)"] \arrow[d, mapsto, "p"'] & e \arrow[d, mapsto, "p"] \\ b' \arrow[r, "f"'] & b \end{tikzcd}$$

$\overline{f}(e)$ p-cartesian

Σ-objects in fam-categories

Def. (Σ-objects) $\mathscr{C}$ fam-category

Σ-objects in fam-categories

Def. (Σ-objects) $\mathscr{C}$ fam-category

$$
\begin{tikzcd}
\sum_c \lambda \arrow[d, "\mathrm{pr}_1^\lambda"] & \\
c \arrow[r, "\lambda"] &
\end{tikzcd}
$$

Σ-objects in fam-categories

Def. (Σ-objects) $\mathscr{C}$ fam-category

$$\begin{tikzcd} & \sum_c \lambda \arrow[d, "\mathrm{pr}_1^\lambda"] & \\ c' \arrow[r, "f"'] & c \arrow[r, "\lambda"] & {} \end{tikzcd}$$

Σ-objects in fam-categories

Def. (Σ-objects) $\mathscr{C}$ fam-category

$$\begin{tikzcd}
\sum_{c'} \lambda \circ f \arrow[r, "\Sigma_\lambda f"] \arrow[d, "\mathrm{pr}_1^{\lambda\circ f}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \sum_c \lambda \arrow[d, "\mathrm{pr}_1^{\lambda}"] & \\
c' \arrow[r, "f"'] & c \arrow[r, "\lambda"] & {}
\end{tikzcd}$$

Σ-objects in fam-categories

Def. (Σ-objects) $\mathscr{C}$ fam-category

$$\begin{tikzcd}
\sum_c \lambda \arrow[r, dotted, "\Sigma_\lambda 1_c"] \arrow[d, "\mathrm{pr}_1^\lambda"'] \arrow[dr, phantom, "\lrcorner", very near start] & \sum_c \lambda \arrow[d, "\mathrm{pr}_1^\lambda"] & \\
c \arrow[r, "1_c"'] & c \arrow[r, "\lambda"] & {}
\end{tikzcd}$$

1. $\sum_\lambda 1_c = 1_{\sum_c \lambda}$

Def. (Σ-objects) $\mathscr{C}$ fam-category

$$\begin{tikzcd}
\sum_{c''}(\lambda\circ f)\circ g \arrow[r, "\Sigma_{\lambda\circ f} g"] \arrow[d, "\mathrm{pr}_1^{(\lambda\circ f)\circ g}"] \arrow[dr, phantom, "\lrcorner", very near start] & \sum_{c'}\lambda\circ f \arrow[r, "\Sigma_\lambda f"] \arrow[d, "\mathrm{pr}_1^{\lambda\circ f}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \sum_c \lambda \arrow[d, "\mathrm{pr}_1^{\lambda}"] & \\
c'' \arrow[r, "g"'] & c' \arrow[r, "f"'] & c \arrow[r, "\lambda"] & {}
\end{tikzcd}$$

1. $\sum_\lambda 1_c = 1_{\sum_c \lambda}$
2. $\sum_\lambda g\circ f = \sum_\lambda g \circ \sum_{g\circ\lambda} f$

Σ-objects in fam-categories

Def. (Σ-objects) $\mathscr{C}$ fam-category

$$\begin{tikzcd}
\sum_{c'} \lambda \circ f \arrow[r, "\Sigma_\lambda f"] \arrow[d, "\mathrm{pr}_1^{\lambda\circ f}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \sum_c \lambda \arrow[d, "\mathrm{pr}_1^{\lambda}"] & \\
c' \arrow[r, "f"'] & c \arrow[r, "\lambda"] & {}
\end{tikzcd}$$

1. $\sum_\lambda 1_c = 1_{\sum_c \lambda}$
2. $\sum_\lambda g \circ f = \sum_\lambda g \circ \sum_{g\circ\lambda} f$

Def. (Discrete comprehension categories) $p \colon \mathscr{E} \to \mathscr{B}$ disc. fibration

Σ-objects in fam-categories

Def. (Σ-objects) $\mathscr{C}$ fam-category

$$
\begin{tikzcd}
\sum_{c'} \lambda \circ f \arrow[r, "\Sigma_\lambda f"] \arrow[d, "\mathsf{pr}_1^{\lambda\circ f}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \sum_c \lambda \arrow[d, "\mathsf{pr}_1^{\lambda}"] & \\
c' \arrow[r, "f"'] & c \arrow[r, "\lambda"] & {}
\end{tikzcd}
$$

1. $\sum_\lambda 1_c = 1_{\sum_c \lambda}$
2. $\sum_\lambda g \circ f = \sum_\lambda g \circ \sum_{g\circ\lambda} f$

Def. (Discrete comprehension categories) $p\colon \mathscr{E} \to \mathscr{B}$ disc. fibration

$$
\begin{tikzcd}
\mathscr{E} \arrow[r, "P"] \arrow[d, "p"] & \mathscr{B}^{[1]} \arrow[dl, "\mathrm{cod}"] \\
\mathscr{B} &
\end{tikzcd}
$$

morph. in $\mathscr{E} \xrightarrow{P}$ cod-cart. morph.

Σ-objects in 2-fam-categories

Def. (Σ-objects) $\mathscr{C}$ (2-)fam-category

$$\begin{tikzcd}
\sum_c \lambda \arrow[dd, "\sum_{\lambda,\mu} \eta"'] \arrow[dr, "\mathrm{pr}_1^\lambda"] & & \\
& c \arrow[r, bend left, "\lambda"] \arrow[r, phantom, "\Downarrow\eta"] \arrow[r, bend right, "\mu"'] & {} \\
\sum_c \mu \arrow[ur, "\mathrm{pr}_1^\mu"'] & &
\end{tikzcd}$$

1. $\sum_{\lambda,\lambda} 1_\lambda = 1_{\sum_c \lambda}$
2. $\sum_{\lambda,\mu} \xi \circ \eta = \sum_{\nu,\mu} \xi \circ \sum_{\lambda,\nu} \eta$

Σ-objects in 2-fam-categories

Def. (Σ-objects) $\mathscr{C}$ (2-)fam-category

$$
\begin{tikzcd}
\sum_c \lambda \arrow[dd, "\sum_{\lambda,\mu} \eta"'] \arrow[dr, "\mathrm{pr}_1^\lambda"] & & \\
& c \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\mu"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \\
\sum_c \mu \arrow[ur, "\mathrm{pr}_1^\mu"'] & &
\end{tikzcd}
$$

1. $\sum_{\lambda,\lambda} 1_\lambda = 1_{\sum_c \lambda}$
2. $\sum_{\lambda,\mu} \xi \circ \eta = \sum_{\nu,\mu} \xi \circ \sum_{\lambda,\nu} \eta$

Σ-objects in 2-fam-categories

Def. (Σ-objects) $\mathscr{C}$ (2-)fam-category

$$\begin{tikzcd}
\sum_c \lambda \arrow[d, "\sum_{\lambda,\mu} \eta"'] \arrow[dr, "\mathrm{pr}_1^\lambda"] & & \\
\sum_c \nu \arrow[r, "\mathrm{pr}_1^\nu"] \arrow[d, "\sum_{\nu,\mu} \xi"'] & c \arrow[r, "\nu" description] \arrow[r, bend left=80, "\lambda"] \arrow[r, bend right=80, "\mu"'] \arrow[r, phantom, shift left=4, "\Downarrow\eta"] \arrow[r, phantom, shift right=4, "\Downarrow\xi"] & {} \\
\sum_c \mu \arrow[ur, "\mathrm{pr}_1^\mu"'] & &
\end{tikzcd}$$

1. $\sum_{\lambda,\lambda} 1_\lambda = 1_{\sum_c \lambda}$
2. $\sum_{\lambda,\mu} \xi \circ \eta = \sum_{\nu,\mu} \xi \circ \sum_{\lambda,\nu} \eta$

Σ-objects in 2-fam-categories

Def. (Σ-objects) $\mathscr{C}$ (2-)fam-category

$$\begin{tikzcd}
\sum_c \lambda \arrow[dd, "\sum_{\lambda,\mu}\eta"'] \arrow[dr, "\mathsf{pr}_1^\lambda"] & & \\
& c \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\mu"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \\
\sum_c \mu \arrow[ur, "\mathsf{pr}_1^\mu"'] & &
\end{tikzcd}$$

1. $\sum_{\lambda,\lambda} 1_\lambda = 1_{\sum_c \lambda}$
2. $\sum_{\lambda,\mu} \xi \circ \eta = \sum_{\nu,\mu} \xi \circ \sum_{\lambda,\nu} \eta$

Def. (Split comprehension categories)
$p\colon \mathscr{E} \to \mathscr{B}$ split fibration

$$\begin{tikzcd}
\mathscr{E} \arrow[r, "P"] \arrow[d, "p"'] & \mathscr{B}^{[1]} \arrow[dl, "\mathrm{cod}"] \\
\mathscr{B} &
\end{tikzcd}$$

p-cart. morph.$\xrightarrow{P}$ cod-cart. morph.

The equivalence

	(2-fam,Σ-categories)	Split comprehension categories
1	category $\mathscr{C}$	base category $\mathscr{B}$
2	family arrow over $c \in \mathscr{C}$	object e over $b \in \mathscr{B}$
3	precomposition $\lambda \circ f$	cartesian lift $\overline{f}(e) : f^*(e) \to e$
4	2-family arrows $\eta : \lambda \to \mu$	vertical arrows $v : e \to e'$
5	$\mathrm{pr}_1^\lambda : \sum_c \lambda$	$P(e) : P_0(e) \to b$
6	$\sum_\lambda f : \sum_c \lambda \circ f \to \sum_c \lambda$	$P_0(\overline{f}(e)) : P_0\big(f^*(e)\big) \to P_0(e)$
7	$\sum_{\lambda,\mu} \eta : \sum_c \lambda \to \sum_c \mu$	$P_0(v) : P_0(e) \to P_0(e')$

6 + 7: here $P_0 := \mathrm{dom} \circ P$

Dependent arrows in fam-categories

$\mathscr{C}$ fam-category

+ collection $\mathrm{dHom}(c, \lambda)$ for every family arrow λ

+ precomposition

1. $\Phi \circ 1_c = \Phi$
2. $\Phi \circ (f \circ g) = (\Phi \circ f) \circ g$

$$c' \xrightarrow[f]{} c \xrightarrow{\overset{\Phi}{\Rightarrow}\ \lambda}$$

Dependent arrows in fam-categories

$\mathscr{C}$ fam-category
+ collection $\mathrm{dHom}(c, \lambda)$ for every family arrow λ
+ precomposition

1. $\Phi \circ 1_c = \Phi$
2. $\Phi \circ (f \circ g) = (\Phi \circ f) \circ g$

$$\begin{tikzcd}
c' \arrow[r, "f"'] \arrow[rr, Rightarrow, bend left, "\Phi\circ f"] & c \arrow[r, "\lambda"] & {}
\end{tikzcd}$$

Dependent arrows in fam-categories

$\mathscr{C}$ fam-category
+ collection $\mathrm{dHom}(c, \lambda)$ for every family arrow λ
+ precomposition

1. $\Phi \circ 1_c = \Phi$
2. $\Phi \circ (f \circ g) = (\Phi \circ f) \circ g$

$$\begin{tikzcd} c' \arrow[r, "f"'] & c \arrow[r, "\lambda"] \arrow[r, phantom, shift left=3, "\overset{\Phi}{\Rightarrow}"] & {} \end{tikzcd}$$

Dependent arrows in 2-fam-categories

$\mathscr{C}$ 2-fam-category

+ collection $\mathrm{dHom}(c, \lambda)$ for every family arrow λ

+ precomposition as before

+ **postcomposition**

  1. $1_\lambda \circ \Phi = \Phi$
  2. $(\xi \circ \eta) \circ \Phi = \xi \circ (\eta \circ \Phi)$

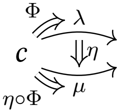
+ **compatibility**

  3. $(\eta \circ \Phi) \circ f = (\eta \circ f) \circ (\Phi \circ f)$

$$\begin{tikzcd} & {} \\ c \arrow[ur, Rightarrow, "\Phi"] \arrow[rr, bend left, "\lambda"] \arrow[rr, bend right, "\mu"'] \arrow[rr, phantom, "\Downarrow\eta"] & & {} \end{tikzcd}$$

Dependent arrows in 2-fam-categories

$\mathscr{C}$ 2-fam-category
+ collection $\mathrm{dHom}(c, \lambda)$ for every family arrow λ
+ precomposition as before
+ **post**composition
  1. $1_\lambda \circ \Phi = \Phi$
  2. $(\xi \circ \eta) \circ \Phi = \xi \circ (\eta \circ \Phi)$
+ **compatibility**
  3. $(\eta \circ \Phi) \circ f = (\eta \circ f) \circ (\Phi \circ f)$



$\mathscr{C}$ 2-fam-category

+ collection $\mathrm{dHom}(c, \lambda)$ for every family arrow λ

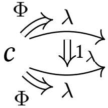
+ precomposition as before

+ **post**composition

  1. $1_\lambda \circ \Phi = \Phi$
  2. $(\xi \circ \eta) \circ \Phi = \xi \circ (\eta \circ \Phi)$

+ **compatibility**

  3. $(\eta \circ \Phi) \circ f = (\eta \circ f) \circ (\Phi \circ f)$

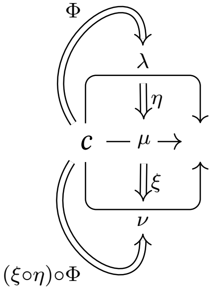


$\mathscr{C}$ 2-fam-category
\+ collection $\mathrm{dHom}(c, \lambda)$ for every family arrow λ
\+ precomposition as before
\+ **post**composition

1. $1_\lambda \circ \Phi = \Phi$
2. $(\xi \circ \eta) \circ \Phi = \xi \circ (\eta \circ \Phi)$

\+ **compatibility**

3. $(\eta \circ \Phi) \circ f = (\eta \circ f) \circ (\Phi \circ f)$



Dependent arrows in 2-fam-categories

$\mathscr{C}$ 2-fam-category
+ collection $\mathrm{dHom}(c, \lambda)$ for every family arrow λ
+ precomposition as before
+ **postcomposition**

  1. $1_\lambda \circ \Phi = \Phi$
  2. $(\xi \circ \eta) \circ \Phi = \xi \circ (\eta \circ \Phi)$

+ **compatibility**

  3. $(\eta \circ \Phi) \circ f = (\eta \circ f) \circ (\Phi \circ f)$

$$\begin{tikzcd} c' \arrow[r, "f"] & c \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\mu"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \end{tikzcd}$$

Dependent arrows in 2-fam-categories

$\mathscr{C}$ 2-fam-category
+ collection $\mathrm{dHom}(c, \lambda)$ for every family arrow λ
+ precomposition as before
+ **postcomposition**

1. $1_\lambda \circ \Phi = \Phi$
2. $(\xi \circ \eta) \circ \Phi = \xi \circ (\eta \circ \Phi)$

+ **compatibility**

3. $(\eta \circ \Phi) \circ f = (\eta \circ f) \circ (\Phi \circ f)$

$$\begin{tikzcd} c' \arrow[r, "f"] & c \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\mu"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \end{tikzcd}$$

Dependent arrows in 2-fam-categories

$\mathscr{C}$ 2-fam-category

+ collection $\mathrm{dHom}(c, \lambda)$ for every family arrow λ

+ precomposition as before

+ **post**composition

1. $1_\lambda \circ \Phi = \Phi$
2. $(\xi \circ \eta) \circ \Phi = \xi \circ (\eta \circ \Phi)$

+ **compatibility**

3. $(\eta \circ \Phi) \circ f = (\eta \circ f) \circ (\Phi \circ f)$

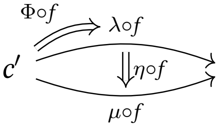
$$\begin{tikzcd} c' \arrow[r, "f"] \arrow[rr, Rightarrow, bend right, "(\eta\circ\Phi)\circ f"'] & c \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\mu"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \end{tikzcd}$$

$\mathscr{C}$ 2-fam-category
+ collection $\mathrm{dHom}(c, \lambda)$ for every family arrow λ
+ precomposition as before
+ **postcomposition**

1. $1_\lambda \circ \Phi = \Phi$
2. $(\xi \circ \eta) \circ \Phi = \xi \circ (\eta \circ \Phi)$

+ **compatibility**

3. $(\eta \circ \Phi) \circ f = (\eta \circ f) \circ (\Phi \circ f)$



Dependent arrows in 2-fam-categories

$\mathscr{C}$ 2-fam-category

+ collection $\mathrm{dHom}(c, \lambda)$ for every family arrow λ

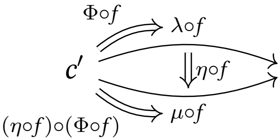
+ precomposition as before

+ **postcomposition**

1. $1_\lambda \circ \Phi = \Phi$
2. $(\xi \circ \eta) \circ \Phi = \xi \circ (\eta \circ \Phi)$

+ **compatibility**

3. $(\eta \circ \Phi) \circ f = (\eta \circ f) \circ (\Phi \circ f)$



Dependent arrows for split fibrations

Given: split fibration
$\rightsquigarrow$ orth. fact. system $(\mathcal{V}, \mathcal{C})$ on $\mathscr{E}$

$$\begin{tikzcd} \mathscr{E} \arrow[d, "p"] \\ \mathscr{B} \end{tikzcd}$$

Dependent arrows for split fibrations

Given: split fibration
$\rightsquigarrow$ orth. fact. system $(\mathcal{V}, \mathcal{C})$ on $\mathscr{E}$
$\rightsquigarrow$ **discrete ambifibration** $q\colon \mathscr{G} \to \mathscr{E}$.

$$\begin{tikzcd} \mathscr{G} \arrow[d, "q"] \\ \mathscr{E} \arrow[d, "p"] \\ \mathscr{B} \end{tikzcd}$$

Dependent arrows for split fibrations

Given: split fibration
$\rightsquigarrow$ orth. fact. system $(\mathcal{V}, \mathcal{C})$ on $\mathscr{E}$
$\rightsquigarrow$ **discrete ambifibration** $q\colon \mathscr{G} \to \mathscr{E}$.

$$\begin{tikzcd} \mathscr{G}_{\mathcal{V}} \arrow[r] \arrow[d, "q_{\mathcal{V}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{G} \arrow[d, "q"] \\ \mathcal{V} \arrow[r] & \mathscr{E} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{G}_{\mathcal{C}} \arrow[r] \arrow[d, "q_{\mathcal{C}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{G} \arrow[d, "q"] \\ \mathcal{C} \arrow[r] & \mathscr{E} \end{tikzcd}$$

$$\begin{tikzcd} \mathscr{G} \arrow[d, "q"] \\ \mathscr{E} \arrow[d, "p"] \\ \mathscr{B} \end{tikzcd}$$

Dependent arrows for split fibrations

Given: split fibration
$\rightsquigarrow$ orth. fact. system $(\mathcal{V}, \mathcal{C})$ on $\mathscr{E}$
$\rightsquigarrow$ **discrete ambifibration** $q \colon \mathscr{G} \to \mathscr{E}$.

$$\begin{tikzcd} \mathscr{G} \arrow[d, "q"] \\ \mathscr{E} \arrow[d, "p"] \\ \mathscr{B} \end{tikzcd}$$

$$\begin{tikzcd} \mathscr{G}_{\mathcal{V}} \arrow[r] \arrow[d, "q_{\mathcal{V}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{G} \arrow[d, "q"] \\ \mathcal{V} \arrow[r] & \mathscr{E} \end{tikzcd} \quad \begin{tikzcd} \mathscr{G}_{\mathcal{C}} \arrow[r] \arrow[d, "q_{\mathcal{C}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{G} \arrow[d, "q"] \\ \mathcal{C} \arrow[r] & \mathscr{E} \end{tikzcd}$$

$\rightsquigarrow$ $p \circ q$ discrete fibration

Dependent arrows and Σ-objects

In Type Theory

$$\begin{tikzcd}
\sum_{x:A} F(x) \arrow[r, "\mathrm{pr}_1"] \arrow[rr, bend right, "F\circ\mathrm{pr}_1"'] & A \arrow[r, "F"] & \mathcal{U}
\end{tikzcd}$$

$$\mathrm{pr}_2 : \prod_{z:\sum_{x:A} F(x)} F\big(\mathrm{pr}_1(z)\big)$$

$\mathscr{C}$ dep-category & (fam,Σ)-category
+ $\mathrm{pr}_2^\lambda \in \mathrm{fHom}(\lambda \circ \mathrm{pr}_1^\lambda)$

$$\begin{tikzcd}
\sum_{c'} \lambda\circ f \arrow[r, "\sum_\lambda f"] \arrow[d, "\mathrm{pr}_1^{\lambda\circ f}"'] & \sum_c \lambda \arrow[d, "\mathrm{pr}_1^\lambda"] \arrow[dr, "\lambda\circ\mathrm{pr}_1^\lambda"] \arrow[r, phantom, "\Rightarrow\ \mathrm{pr}_2^\lambda"] & {} \\
c' \arrow[r, "f"'] & c \arrow[r, "\lambda"] & {}
\end{tikzcd}$$

s.t. $\mathrm{pr}_2^\lambda \circ \sum_\lambda f = \mathrm{pr}_2^{\lambda\circ f}$

Dependent arrows and Σ-objects

In Type Theory

$$\begin{tikzcd}
\sum_{x:A} F(x) \arrow[r, "\mathrm{pr}_1"] \arrow[rr, bend right, "F\circ\mathrm{pr}_1"'] & A \arrow[r, "F"] & \mathcal{U}
\end{tikzcd}$$

$$\mathrm{pr}_2 : \prod_{z:\sum_{x:A} F(x)} F\big(\mathrm{pr}_1(z)\big)$$

$\mathscr{C}$ dep-category & (fam,Σ)-category
+ $\mathrm{pr}_2^\lambda \in \mathrm{fHom}(\lambda \circ \mathrm{pr}_1^\lambda)$

$$\begin{tikzcd}
\sum_{c'} \lambda \circ f \arrow[r, "\sum_\lambda f"] \arrow[d, "\mathrm{pr}_1^{\lambda\circ f}"'] & \sum_c \lambda \arrow[d, "\mathrm{pr}_1^\lambda"] \arrow[dr, "\lambda\circ\mathrm{pr}_1^\lambda"] & \\
c' \arrow[r, "f"'] & c \arrow[r, "\lambda"] & {}
\end{tikzcd}$$

s.t. $\mathrm{pr}_2^\lambda \circ \sum_\lambda f = \mathrm{pr}_2^{\lambda\circ f}$

A small remark

Given a comprehension category

$$\begin{tikzcd} \mathscr{E} \arrow[r, "P"] \arrow[d, "p"] & \mathscr{B}^{[1]} \arrow[dl, "\mathrm{cod}"] \\ \mathscr{B} & \end{tikzcd} \quad \leftrightarrow \quad \begin{tikzcd} \mathscr{E} \arrow[d, shift right=4, "P_0"'] \arrow[d, shift left=4, "p"] \arrow[d, phantom, "\overset{P}{\Rightarrow}"] \\ \mathscr{B} \end{tikzcd}$$

via $P_e := P(e) \colon P_0(e) \to p(e)$

Higher discrete comprehension categories

Def. start with $P \colon P_0 \Rightarrow p$.

$$\begin{tikzcd} \mathscr{E} \arrow[d, shift right=4ex, "P_0"'] \arrow[d, shift left=4ex, "p"] \arrow[d, phantom, "\overset{P}{\Rightarrow}"] \\ \mathscr{B} \end{tikzcd}$$

Higher discrete comprehension categories

Def. start with $P\colon P_0 \Rightarrow p$.
+ $q\colon \mathscr{H} \to \mathscr{E}$ discrete fibration

$$\begin{tikzcd} \mathscr{H} \arrow[d, "q"] \\ \mathscr{E} \arrow[d, bend right, "P_0"'] \arrow[d, bend left, "p"] \arrow[d, phantom, "\overset{P}{\Rightarrow}"] \\ \mathscr{B} \end{tikzcd}$$

Higher discrete comprehension categories

Def. start with $P \colon P_0 \Rightarrow p$.
+ $q \colon \mathscr{H} \to \mathscr{E}$ discrete fibration
+ $\mathrm{pr}_2 \colon \mathscr{E} \to \mathscr{H}$

$$\begin{tikzcd}
\mathscr{H} \arrow[d, shift left=2ex, "q"] \\
\mathscr{E} \arrow[u, shift left=2ex, "\mathrm{pr}_2"] \arrow[d, shift right=2ex, "P_0"'] \arrow[d, shift left=2ex, "p"] \arrow[d, phantom, "\overset{P}{\Rightarrow}"] \\
\mathscr{B}
\end{tikzcd}$$

Higher discrete comprehension categories

Def. start with $P\colon P_0 \Rightarrow p$.
+ $q\colon \mathscr{H} \to \mathscr{E}$ discrete fibration
+ $\mathrm{pr}_2\colon \mathscr{E} \to \mathscr{H}$
+ $\xi\colon q \circ \mathrm{pr}_2 \Rightarrow \mathrm{id}_{\mathscr{E}}$ s.t. $p * \xi = P$

$$\begin{tikzcd}
\mathscr{H} \arrow[d, shift left=3, "q"] \\
\mathscr{E} \arrow[u, shift left=3, "\mathrm{pr}_2"] \arrow[u, phantom, shift left=0, "\Downarrow\xi"] \arrow[d, shift right=3, "P_0"'] \arrow[d, shift left=3, "p"] \arrow[d, phantom, "\overset{P}{\Rightarrow}"] \\
\mathscr{B}
\end{tikzcd}$$

The equivalence

	(dep,Σ)-categories	Higher discrete comprehension categories
1	dependent arrows Φ	objects of $\mathscr{H}$
2	precomposition $\Phi \circ f$	$\overline{p \circ q}(\Phi)$
3	pr_2^λ	$\mathsf{pr}_2(\lambda)$
4	$\mathsf{pr}_2^\lambda \in \mathrm{dHom}(\lambda \circ \mathsf{pr}_1^\lambda)$	$p * \chi = P$
5	$\mathsf{pr}_2^\lambda \circ \sum_\lambda f = \mathsf{pr}_2^{\lambda \circ f}$	$p * \chi = P$

Outlook

1. Find the missing equivalence

Outlook

1. Find the missing equivalence
2. Comparison to generalised categories with attributes

Outlook

1. Find the missing equivalence
2. Comparison to generalised categories with attributes
3. Examination of dualised structure

$$
\begin{tikzcd}
c' \arrow[r, "f"'] & c \arrow[r, "\lambda"] \arrow[r, phantom, bend left, "\Phi"] & {}
\end{tikzcd}
$$

$$
\to \quad
\begin{tikzcd}
{} \arrow[r, "\lambda"] \arrow[r, phantom, bend left, "\Phi"] & c' \arrow[r, "f"'] & c
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathrm{co}(\mathrm{dep},\Sigma)\mathrm{C} \arrow[rr] & & \mathrm{co}(2\text{-}\mathrm{dep},\Sigma)\mathrm{C} \\
\mathrm{codepC} \arrow[ur] \arrow[rr] & & \mathrm{co2\text{-}depC} \arrow[ur] & \\
& \mathrm{co}(\mathrm{fam},\Sigma)\mathrm{C} \arrow[uu] \arrow[rr] & & \mathrm{co}(2\text{-}\mathrm{fam},\Sigma)\mathrm{C} \arrow[uu] \\
\mathrm{cofamC} \arrow[uu] \arrow[ur] \arrow[rr] & & \mathrm{co2\text{-}famC}\,. \arrow[uu] \arrow[ur] &
\end{tikzcd}
$$