Realizability models for set theory

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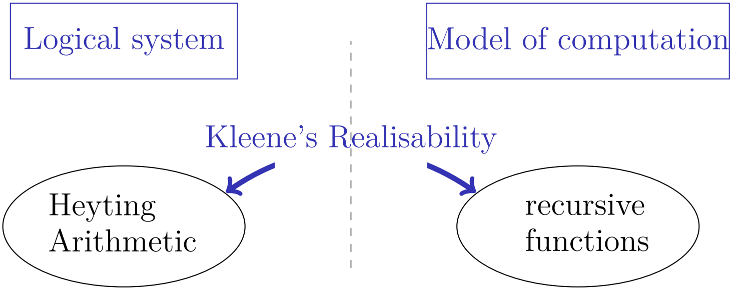
Realizability

Realizability: invented by Kleene in 1945, the goal of realizability is to interpret mathematical theories in some model of computation in order to extract the computational content of mathematical proofs.

We establish a correspondence between formulas and programs in a way that is compatible with the rules of deduction.

From Curry Howard to Forcing



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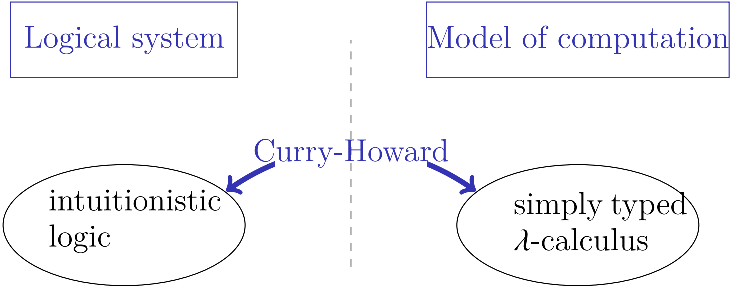


From Curry Howard to Forcing

Logical system

Model of computation

Griffin

classical logic

λ_c-calculus

From Curry Howard to Forcing

Logical system

Model of computation

Krivine

set theory

enriched
λ_c-calculus

Forcing in a nutshell

Cohen 1963: Forcing is the main technique for defining models of ZF(C) and proving independence and relative consistency results.

We consider a Boolean algebra $(\mathbb{B}, 0, 1, \wedge_{\mathbb{B}} \vee_{\mathbb{B}}, \neg_{\mathbb{B}})$, and we "evaluate" each formula of set theory using the elements of $\mathbb{B}$:

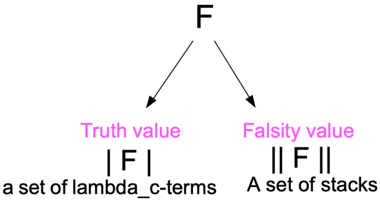
$$\varphi \mapsto \|\varphi\| \in \mathbb{B}$$

$$\|\varphi \wedge \psi\| = \|\varphi\| \wedge_{\mathbb{B}} \|\psi\|$$
$$\|\varphi \vee \psi\| = \|\varphi\| \vee_{\mathbb{B}} \|\psi\|$$
$$\|\neg\varphi\| = \neg_{\mathbb{B}}\|\varphi\|$$

$$\mathrm{p} \Vdash \varphi \text{ ("p forces } \varphi\text{") when } \mathrm{p} \leq \|\varphi\|$$

Given an ultrafilter U on $\mathbb{B}$, the Theory $:= \{\varphi : \exists \mathrm{p} \in \mathrm{U}(\mathrm{p} \Vdash \varphi)\}$ forms a coherent classical theory which contains ZF(C) (if we do the process starting from a model of ZF(C), the ground model)

Classical realizability in a nutshell



A program p $\Vdash$ F (“p realizes F”) when it is in the truth value of F, that is when p is ‘orthogonal’ to every stack π in the falsity value of F,

meaning that the process $p * \pi \in \perp\!\!\!\perp$, where $\perp\!\!\!\perp$ is the so-called ‘Pole’.

Classical realizability in a nutshell

Then we choose some privileged λ_c-terms that we call realizers/proof-like terms, here denoted $\mathcal{PL}$.

$$\mathrm{T} = \{\varphi | \ \exists\theta \in \mathcal{PL}(\theta \Vdash |\varphi|)\}$$

Classical realizability in a nutshell

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We show that T forms a coherent theory, which is closed by classical natural deduction and under the right constraints it will contain the axioms we want to realize (e.g. ZF); a realizability model is a model of such a theory.

Realizability algebras

The main ingredients are :

- Λ a set of programs (λ_c-terms+ ...)
- Π a set of stacks
- $\mathcal{PL}$ a set of realizers/proof-like terms
- $\prec_K$ the execution, a pre-order on processes $t * \pi$ (where t is a program and π is a stack)
- $\perp\!\!\!\perp$ the pole a set of processes $\prec_K$-upward closed (it defines the "orthogonal processes")

Terms and stacks

$\Lambda^{\text{open}}_{A,B}$ (λ_c-terms) :

$$
\begin{array}{rcll}
t, u & ::= & x & \text{(variable; we choose a countable set of variables} \\
& | & tu & \text{(application)} \\
& | & \lambda x.t & \text{(abstraction; x is a variable and t is a } \lambda_c\text{-term)} \\
& | & cc & \text{(call-with-current-continuation)} \\
& | & k_\pi & \text{(continuation constants; } \pi \text{ is a stack)} \\
& | & \xi_\alpha & \text{(special terms; } \alpha \in A\text{)}
\end{array}
$$

$\Pi_{A,B}$ (Stacks) :

$$
\begin{array}{rcll}
\pi & ::= & \omega_\beta & \text{(stack bottoms; } \beta \in B\text{)} \\
& | & t \bullet \pi & \text{(t is a closed } \lambda_c\text{-term and } \pi \text{ is a stack)}
\end{array}
$$

The execution

$\prec_K$ is the smallest preorder on the set of processes such that

$$\begin{array}{rcll} tu * \pi & \succ_K & t * u \bullet \pi & \text{(push)} \\ \lambda x.t * u \bullet \pi & \succ_K & t[x := u] * \pi & \text{(grab)} \\ cc * t \bullet \pi & \succ_K & t * k_\pi \bullet \pi & \text{(save)} \\ k_{\pi'} * t \bullet \pi & \succ_K & t * \pi' & \text{(restore)} \end{array}$$

Classical realizability

No stack is in $\|\top\|$, every stack is in $\|\bot\|$

$\|\varphi \Rightarrow \psi\| = \{\xi \,.\, \pi;\ \xi \in |\varphi| \text{ and } \pi \in \|\psi\|\}$

$\xi \Vdash \varphi$ (also denoted $\xi \in |\varphi|$) if $\xi * \pi \in \perp\!\!\!\perp$ for every stack in $\|\varphi\|$

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call-cc realizes Peirce's law $(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$

Non extensional set theory

In order to realize the axioms of set theory, we work with a non-extensional version of ZF, called ZF_ε, with...

two membership relations:

- $\in$ the usual extensional one
- $\boldsymbol{\varepsilon}$ a (strict) non-extensional one

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two membership relations:

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... and two equality relations

- $\simeq$ the usual extensional one
- $=$ Leibniz identity

The realizability relation

We define $|\varphi|$ and $\|\varphi\|$ simultaneously and by induction on φ in the language of realizability.

- no stack is in $\|\top\|$, every stack is in $\|\bot\|$
- $\|a \not\varepsilon b\| = \{\pi \text{ stack} \; ; \; (a, \pi) \in b\}$
- $\|a \subseteq b\| = \{\xi \,.\, \pi ; \; \exists c (c, \pi) \in a \text{ and } \xi \Vdash c \notin b\}$
- $\|a \notin b\| = \{\xi \,.\, \xi' \,.\, \pi ; \; \exists c (c, \pi) \in b \text{ and } \xi \Vdash a \subseteq c \text{ and } \xi' \Vdash c \subseteq a\}$
- $\|\varphi \Rightarrow \psi\| = \{\xi \,.\, \pi ; \; \xi \Vdash \varphi \text{ and } \pi \in \|\psi\|\}$
- $\|\forall x \varphi\| = \bigcup_a \|\varphi[a/x]\|$

$$\xi \in |\varphi| \iff \forall \pi \in \|\varphi\| (\xi \star \pi \in \perp\!\!\!\perp)$$

$\xi \Vdash \varphi$ means $\xi \in |\varphi|$

Realizability models for set theory

ZF_ε is a conservative extension of ZF. The realizability theory contains the axioms of ZF_ε (provided ZF is consistent).

Any model of such a theory yields a structure (actually many) in the language of ZF_ε denoted $\mathcal{N}_\varepsilon$ or $\mathcal{N}$, and a structure (actually many) in the language of ZF, denoted $\mathcal{N}_\in$.

Realizability models for set theory

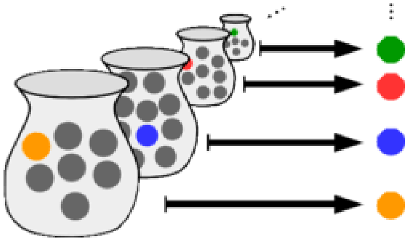
Thus the axioms of set theory are realizable.

Exemple:

- the Pairing axiom is realized by λx.((xI)I),
- the Axiom of Foundation is realized by the fixed point combinator,
- Dependent choice can be realized by quote/a clock...

The Axiom of Choice

What is the computational content
of the axiom of choice?

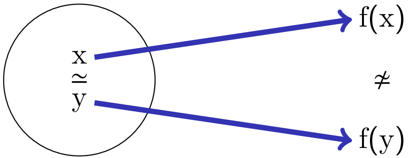


Non extensional axiom of choice

We can "easily" realize a non extensional version of AC, called NEAC

NEAC: existence of a non-extensional choice function, i.e.

$$
\begin{array}{lcl}
x = y & \Rightarrow & f(x) = f(y), \text{ but} \\
x \simeq y & \not\Rightarrow & f(x) \simeq f(y)
\end{array}
$$



Realizing dependent choice

Krivine 2004-2016

Dependent Choice can be realized using quote/a clock and or the bar recursion.

$$\mathrm{quote} * \mathrm{t}.\mathrm{s}.\pi \succ \mathrm{t} * \underline{\mathrm{n}}_{\mathrm{s}}.\pi$$

where $\mathrm{s} \mapsto \mathrm{n_s}$ is some fixed enumeration of Λ in order-type ω.

Realizing the axiom of choice

Fontanella Geoffroy 2021

For every cardinal κ in a model of ZFC, we can construct a realizability model of $\mathrm{ZF} + \mathrm{ZL}_{\hat{\kappa}}$ using a “generalized quote”.

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Fontanella Furlan 2025

For every cardinal κ in a model of ZFC, we can construct a realizability model of $\mathrm{ZF} + \mathrm{AC}_{\hat{\kappa}} + \mathrm{CH}_{\hat{\kappa}}$ using a "generalized bar induction".

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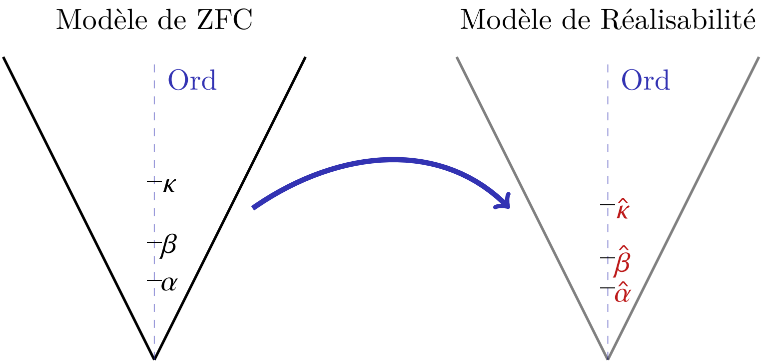
Krivine 2021

The full Axiom of Choice can be realized (but no explicit realizer).

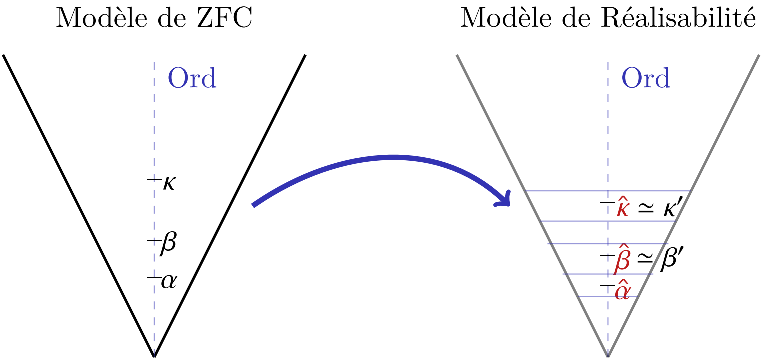
Hat ordinals - Fontanella Geoffroy 2021



Fontanella Geoffroy 2021

Starting from a model $\mathcal{M}$ of ZF + global choice we can define for every cardinal κ of $\mathcal{M}$ a realizability model where κ has a representative $\hat{\kappa}$ such that $\mathrm{ZF} + \mathrm{ZL}_{\hat{\kappa}}$ is realized

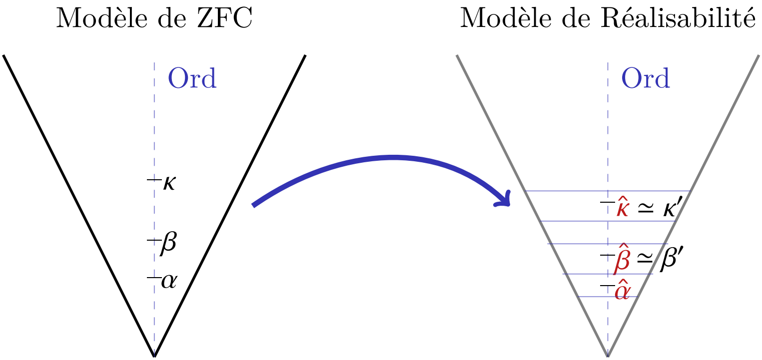
Hat ordinals - Fontanella Geoffroy 2021



Sketch - the representatives

- We consider a calculus with κ many terms
- We add an instruction for comparing terms (a generalized quote)
- we define for every ordinal $\alpha \leq \kappa$ in the ground model, a set $\hat{\alpha}$
- We show that "hat ordinals" "represent" their counterparts

Hat ordinals - Fontanella Geoffroy 2021



Sketch - realizing $ZL_{\hat{\kappa}}$

- $\hat{\kappa}$ has a $=$-unique representative of each of its $\simeq$-classes of elements
- NEAC entails a choice function over the representatives
- we assign the same value to the other elements in the same class
- we realize $ZL_{\hat{\kappa}}$

Hat ordinals - Fontanella Geoffroy 2021

We fix an enumeration of the terms $\Lambda = \{\nu_\alpha\}_{\alpha<\kappa}$.

For every $\alpha \leq \kappa$,

$$\hat{\alpha} := \{(\hat{\beta}, \nu_\beta \bullet \pi) | \pi \in \Pi, \beta < \alpha\}$$

Proposition

For every $\alpha \leq \kappa$, we can realize (by a proof-like term) that $\hat{\alpha}$ is an ε-ordinal.

Over ZF, equivalent definitions of ordinals:

1. An ordinal is a transitive set which is well-ordered by $\in$.
2. An ordinal is a transitive set a satisfying trichotomy. That is, for any $x, y, \in a$, precisely one of the following hold: $x \in y$, $y \in x$, or $x = y$.
3. An ordinal is a transitive set of transitive sets.

Fontanella Matthews 2025

In ZF_ε these notions are not equivalent.

1. an ε-transitive set which is ε-well-ordered.
2. an ε-TOD : an ε-transitive set a satisfying ε-trichotomy.
3. an ε-ordinal: an ε-transitive set of ε-transitive sets

The first two are equivalent and stronger. However ε-ordinal implies $\in$-ordinal.

Gimel and Reish operators

Gimel function:

$$\gimel x = x \times \Pi$$

Reish function (recursive):

$$ר x = \{(ר y, \pi);\ y \in x, \pi \in \Pi\}$$

Gimel and Reish operators

Gimel function:

$$\gimel x = x \times \Pi$$

Reish function (recursive):

$$\daleth x = \{(\daleth y, \pi);\ y \in x, \pi \in \Pi\}$$

In non trivial realizability models (i.e. non-forcing models), $\daleth 2$ has more than two ε-elements (4, infinitely many...).

Recursive ordinals

Fontanella Matthews APAL 2025

- $\urcorner\alpha$ and $\hat{\alpha}$ are always ε-ordinals

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- $\urcorner\alpha$ and $\hat{\alpha}$ are always $\boldsymbol{\varepsilon}$-ordinals
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- recursive ordinals are usually not ε-TODs

Preserving cardinals

In Forcing and in classical realizability cardinals may 'collapse' (ex. κ may be an uncountable cardinal in the ground model, but $\hat{\kappa}$ may become a countable ordinal in the realizability model).

- cardinals strictly greater than the size of the forcing are preserved
- a forcing with the κ-chain condition preserves κ
- a forcing with the κ-closure property preserves all cardinals $< \kappa$

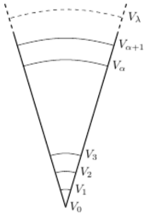
Preserving cardinals

- Fontanella Matthews 2025 : cardinals strictly greater than the size of the realizability algebra are preserved
- Fontanella Geoffroy 2021 + Fontanella Matthews 2025 : a κ-chain condition for realizability agebras to preserve $\hat{\kappa}$ and $\daleth\kappa$
- Krivine 2016 : an $\aleph_1$-closure property for realizability algebras
- Fontanella Furlan 2025 : a κ-closure property for realizability algebras to preserve all cardinals $< \hat{\kappa}$ (Fontanella Furlan 2025)

Large cardinals

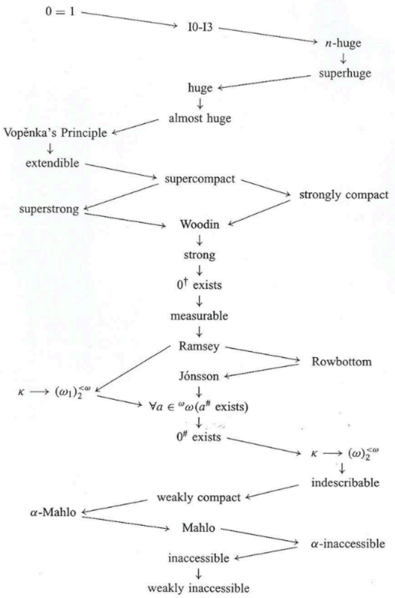
Large cardinals axioms are strong axioms of infinity that assert the existence of uncountable cardinals with various closure properties.

Large cardinals axioms can be ordered by their consistency strength and they all entail the existence of set-models of ZF.



.... hence (by Gödel's second incompleteness theorem) the existence of large cardinals cannot be proven within ZF.

The hierarchy of large cardinals



Realizing Large Cardinals

Fontanella Geoffroy Matthews CSL2024

The axioms of inaccessible, Mahlo, measurable and Reinhardt cardinals are preserved by realizability algebras smaller than the cardinals considered.

The work of H. Friedman and A. Ščedrov provided a suitable formulation of large cardinals in intuitionistic set theory.

Realizability models for inaccessible cardinals

An uncountable cardinal κ is strongly inaccessible if it is not a sum of fewer than κ cardinals smaller than κ, and $\gamma < \kappa$ implies $2^\gamma < \kappa$.
The existence of a strongly inaccessible cardinal is equivalent to V_κ being a Grothendieck universe containing ω.

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Fontanella, Geoffroy, Matthews CSL2024

Let V be a model of ZFC with an inaccessible cardinal κ, and $\mathcal{A} \in V_\kappa$ a realizability algebra. Let $\mathcal{N}$ be the corresponding realizability model, then $\mathcal{N} \Vdash$ ZF + there exists a Grothendieck universe that contains ω.

idea of the proof

- We realize that $\daleth(V_\kappa)$ is an ε-Grothendieck Universe
- begin an ε-Grothendieck universe implies being a $\in$-Grothendieck universe.

Realizability models for Reinhardt cardinals

A Reinhardt cardinal is the critical point of a non-trivial elementary embedding of the universe of sets into itself $j : V \to V$, namely a class function $j : V \to V$ where V is the universe of sets, j is not the identity and for every formula φ and sets $a_1, ..., a_n$ we have

$$\varphi(a_1, .., a_n) \iff \varphi(j(a_1), ..., j(a_n))$$

κ is the critical point means that $j \restriction \kappa = \mathrm{id}$ and $j(\kappa) > \kappa$.

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The existence of Reinhardt cardinals is inconsistent with ZFC by Kunen's inconsistency theorem, so they are defined only in the context of ZF.

Realizability models for Reinhardt cardinals

Fontanella, Geoffroy, Matthews CSL2024

Let κ be a Reinhardt cardinal and $\mathcal{A} \in V_\kappa$ a realizability algebra. Let $\mathcal{N}$ be the corresponding realizability model, then
$\mathcal{N} \Vdash \mathrm{ZF} +$ there exists a Reinhardt cardinal

idea of the proof

- Let $j : V \to V$ be an elementary embedding with critical point κ.
- j induces an elementary embedding j^* of $\mathcal{N}$ onto $\mathcal{N}$,
- $j^*(\daleth(\kappa)) \neq \daleth(\kappa)$
- hence j^* is non trivial and $\daleth(\kappa)$ is a Reinhardt cardinal

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- all large cardinals axioms can be realized with this technique, but these four large cardinals axioms are “preserved” by small realizability algebras
- realizing large cardinals axioms does not require special terms

Some open problems

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  Karagila's conjecture: classical realizability models are symmetric extensions.

Thank you