Universes in synthetic $(\infty, 1)$-category theory

Daniel Gratzer[a] **Jonathan Weinberger**[b][1] Ulrik Buchholtz[c]
Nikolai Kudasov[d] Emily Riehl[e]

[a]Aarhus University, Denmark, [b]**Chapman University, USA**, [c]University of Nottingham, United Kingdom,
[d]Innopolis University, Russia, [e]Johns Hopkins University, USA




Synthetic mathematics, logic-affine computation and efficient proof systems,
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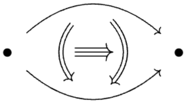
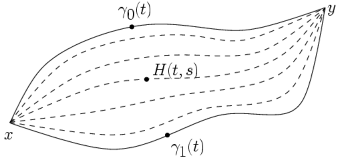
Dedicated to the dear memory of
Thomas Streicher (1958–2025)

∞-category theory

∞-categories:

- **higher morphisms** and **weak composition**





Created with xfig by Yonatan,
`https://commons.wikimedia.org/wiki/File:Homotopy_curves.png`

- Applications: Derived algebraic geometry, stable homotopy theory, topological quantum field theories, higher rewriting, ...
- Dream: $(\infty, 1)$-category theory should be like 1-category theory, but up-to-homotopy.
- Problems in **analytic**, set-theoretic foundations: heavy encoding and not homotopy-invariant. Can we do better?
- **Yes, in synthetic foundations, and using the power of lattice theory and modal logic!**

Functoriality and naturality for free! Reduction to finite-dimensional arguments
Proofs less model-dependent Verification via computer
"∞-category theory for undergraduates"(!?) [Rie23]

Simplicial homotopy type theory

- **Question:** How to extend HoTT to capture ∞-categories?
- **Answer:** (after Riehl–Shulman)**:** HoTT + interval $\mathbb{2} :\equiv \Delta^1$
- Recall Emily's lecture:

$$\Delta^1 = \begin{tikzcd} \bullet \arrow[d] \\ \bullet \end{tikzcd} \qquad \Delta^2 = \begin{tikzcd} \bullet \arrow[r] \arrow[dr] & \bullet \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\ & \bullet \end{tikzcd} \qquad \Lambda^2_1 = \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[d] \\ & \bullet \end{tikzcd} \qquad \mathbb{E} = \begin{tikzcd} \bullet \arrow[d] \arrow[dr, equal] & \\ \bullet \arrow[r] \arrow[ur, phantom, "\Nearrow"] \arrow[dr, equal] & \bullet \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\ & \bullet \end{tikzcd}$$


Higher Structures 1(1):116–193, 2017.

HIGHER STRUCTURES

A type theory for synthetic ∞-categories

Emily Riehl[a] and Michael Shulman[b]

[a] Dept. of Mathematics, Johns Hopkins U., 3400 N Charles St., Baltimore, MD 21218
[b] Dept. of Mathematics, University of San Diego, 5998 Alcalá Park, San Diego, CA 92110

Synthetic $(\infty, 1)$-category theory

Definition (Synthetic ∞-categories [RS17])

A type A is . . .

- **Segal** or a **pre-∞-category** if $A^{\Delta^2} \simeq A^{\Lambda^2_1}$:

$$\{\blacktriangle\} \simeq \{\wedge\}$$

- **Rezk** or a ∞**-category** if it is Segal and $A \simeq A^{\mathbb{E}}$:

$$\{\bullet\} \simeq \{\bullet \cong \bullet\}$$

- an ∞**-groupoid** if $A^{\mathbb{2}} \simeq A$:

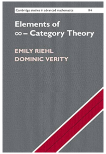
$$\{\bullet\} \simeq \{\bullet \to \bullet\}$$

Definition (hom type)

The **hom type** for $a, b : A$ is: $$\hom_A(a, b) := \sum_{f : \mathbb{2} \to A} f(0) = a \times f(1) = b$$

Models and other synthetic approaches

- Just as HoTT is the internal language of ∞-toposes $\mathcal{E}$ (Awodey's conjecture solved by [Shu19]), sHoTT is the internal language of simplicial objects $\mathcal{E}^{\Delta^{\mathrm{op}}}$; cf. El Mehdi's talk.
- Models will be (internal) complete Segal spaces, cf. Martini and Wolf's internal ∞-topos theory [MW23].
- Rasekh [Ras25] constructs nonstandard models (as filter quotients).
- There are also close parallels to Riehl–Verity's ∞-cosmos theory [RV22] and Cisinski–Cnossen–Nguyen–Walde's synthetic $(\infty, 1)$-category theory [Cno25].
- Some of our structure appears also in d'Espalungue's internal approach to hierarchies of higher structures [dAr23], cf. Sophie's talk.

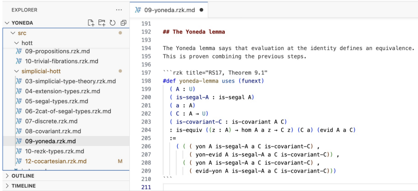


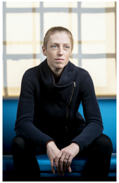
Previous work in sHoTT

- Basic CT, fibered Yoneda lemma, adjunctions [RS17]
- Limits and colimits [Bar22]
- Cartesian fibrations and generalizations [BW23; Wei24a; Wei24b] (cf. also [RV22])
- Conduché fibrations [Bar24]
- **Today:** work in an extended modal framework, giving rise to:
- directed univalent universe $\mathcal{S}$ for ∞-groupoids and left fibrations [GWB24] (using *modal operators*), cf. [Rie18; WL20; Wea24]
- functorial Yoneda lemma for presheaves valued in $\mathcal{S}$ [GWB25]
- cocompleteness, spectra, and generalized cohomology, *in preparation*
- the synthetic ∞-category Cat of ∞-categories, *in preparation*

Formalizing ∞-categories in Rzk

- Kudasov has developed the Rzk proof assistant, implementing sHoTT: `https://rzk-lang.github.io/`
- Using Rzk we initiated the first ever formalizations of ∞-category theory.
- In spring 2023, with Kudasov and Riehl we formalized the (discrete fibered) Yoneda lemma of ∞-category theory: `https://emilyriehl.github.io/yoneda/`
- alongside many other results
- Many proofs in this ∞-dimensional setting *easier* than in dimension 1!
- Formalization helped find a mistake in original paper
- More students & researchers joined us developing a library for ∞-category theory: `https://rzk-lang.github.io/sHoTT/` **Join us!**





Fibered Yoneda Lemma

Theorem (Yoneda Lemma for covariant families (Riehl–Shulman))

Let A be a Segal type, and $a : A$ any term. For a covariant type family $C : A \to \mathcal{U}$, evaluation at id_a is an equivalence:

$$\mathrm{evid}_a^C : \left(\prod_{x:A} \hom_A(a,x) \to C(x) \right) \xrightarrow{\simeq} C(a)$$

- The inverse map is given by

$$\mathbf{y}_a^C : C(a) \to \left(\prod_{x:A} \hom_A(a,x) \to C(x) \right), \quad \mathbf{y}_a^C(u)(x)(f) :\equiv f_! u$$

- Proof “simply” follows from naturality properties and covariance of $\hom_A(a,-)$.
- There also exists a *dependent version* giving **directed path induction**.
- Both have been formalized in Kudasov’s new proof assistant Rzk in [Kud23].
- More general but **easier** than (analytic) 1-categorical proof!

Simplicial vs cubical models

Simplex category $\Delta = \{[n] \mid n \geq 0\}$, **Cube category** $\square = \{[1]^n \mid n \geq 0\}$

Theorem ([KV20; Sat19; SW21])

Simplicial spaces $[\Delta^{\mathrm{op}}, \mathcal{S}]$ *are an essential sub-*∞*-topos of cubical spaces* $[\square^{\mathrm{op}}, \mathcal{S}]$

Internally, a cubical type A is **simplicial** if

$$\mathrm{isSimp}(A) :\equiv \prod_{i,j:\mathbb{I}} \mathrm{isEquiv}(A^{!} : A \to A^{i \leq j \vee j \leq i}).$$

This defines a lex **modality** à la [RSS20]; see also [Wil25].
We need the outer cubical layer to define the correct (categorical) universes (using [Lic+18; Ril24]).

Multimodal type theory

Organize categorical operations via **multimodal type theory (MTT)** [Gra+20; Gra23b; Shu23]. MTT is parametrized by a **mode theory** $\mathcal{M}$, i.e. a 2-category with:

- objects m (**modes**)
- morphisms $\mu : m \to n$ (**modalities**)
- 2-cells $\mu : m \to n$ (**natural maps of modalities**)

A model is given by a 2-functor $F : \mathcal{M} \to \mathsf{Cat}$, with each F_μ lcc, and such that there are left adjoints $L_\mu \dashv F_\mu$.

Theorem (Normalization for MTT [Gra23a])

If $\mathcal{M}$ is decidable, then MTT has normalization and decidable type-checking.

Actions on contexts and types: Interpret Γ/μ via L_μ and $\langle \mu \mid A \rangle$ via F_μ.
Formation and introduction:

$$\frac{\Gamma/\mu \vdash A @ m}{\Gamma \vdash \langle \mu \mid A \rangle @ n \qquad \Gamma, x :_\mu A \;\; \mathsf{ctx} @ n}\;\mu\text{-F} \qquad\qquad \frac{\Gamma/\mu \vdash M : A @ m}{\Gamma \vdash \mathsf{mod}_\mu(M) : \langle \mu \mid A \rangle @ n}\;\mu\text{-I}$$

MTT: Variable and elimination rules

The **variable** rule comes from the counit:

$$\frac{\mu : m \to n}{\Gamma, x :_\mu A/\mu \vdash x : A@m}\ \mu\text{-var}$$

For arbitrary 2-cells we get more generally:

$$\frac{\mu, \nu : m \to n \qquad \alpha : \mu \Rightarrow \nu}{\Gamma, x :_\mu A/\mu \vdash x^\alpha : A^\alpha@m}\ \alpha\text{-var}$$

Weakening:

$$\frac{\mu : m \to n \qquad \alpha : \mu \Rightarrow \mathsf{mods}(\Gamma')}{\Gamma, x :_\mu A, \Gamma' \vdash x^\alpha : A^\alpha@m}\ \alpha\text{-wk}$$

Elimination is given by pattern matching:

$$\frac{\begin{array}{c}\mu : m \to n \quad \nu : n \to o \\ \Gamma, x :_\nu \langle \mu \mid A\rangle \vdash B@o \\ \Gamma/\nu \vdash M_0 : \langle \mu \mid A\rangle@n \\ \Gamma, y :_{\nu\mu} A \vdash M_1 : B[\mathsf{mod}_\mu(y)/x]@o\end{array}}{\Gamma \vdash \mathsf{let}_\nu\ \mathsf{mod}_\mu(x) \leftarrow M_0\ \mathsf{in}\ M_1 : B[M_0/x]@o}\ \mu,\nu\text{-E}$$

From these we can derive **coercion**

$$\mathsf{coe}_\alpha : \langle \mu \mid A\rangle \to \langle \nu \mid A\rangle$$

and **composition**:

$$\mathsf{comp} : \langle \mu\nu \mid A\rangle \to \langle \mu \mid \langle \nu \mid A\rangle\rangle$$

Triangulated type theory

Triangulated type theory $\mathrm{TT}_{\boxslash}$ is sHoTT + MTT with single mode m and the following mode morphisms:

- **Opposite** op: $\langle \mathrm{op} \mid A \rangle$ has its n-simplices reversed
- **Discretization/core** $\flat$: $\langle \flat \mid A \rangle \to A$ is the maximal subgroupoid of A
- **Codiscretization** $\sharp$: $A \to \langle \sharp \mid A \rangle$ is localization at $\partial\Delta^n \to \Delta^n$ (for crisp closed types)
- **Twisted arrows** tw: $\langle \mathrm{tw} \mid A \rangle$ has as n-simplices:

$$\begin{tikzcd}
a_n \arrow[d] & \dots \arrow[l] & a_2 \arrow[l] & a_1 \arrow[l] & a_0 \arrow[l] \\
a_{n+1} \arrow[r] & \dots \arrow[r] & a_{2n-2} \arrow[r] & a_{2n-1} \arrow[r] & a_{2n}
\end{tikzcd}$$

Mode theory:

$$\flat \circ \flat = \flat \circ \mathrm{op} = \flat \circ \sharp = \mathrm{tw} \circ \flat = \mathrm{op} \circ \flat = \flat \qquad \sharp \circ \sharp = \sharp \circ \flat = \sharp \circ \mathrm{op} = \mathrm{op} \circ \sharp = \sharp$$

$$\mathrm{op} \circ \mathrm{op} = \mathrm{id} \qquad \varepsilon : \flat \to \mathrm{id} \qquad \eta : \mathrm{id} \to \sharp$$

$$\eta \cdot \sharp = \sharp \cdot \eta = \mathrm{id} : \sharp \to \sharp \qquad \flat \cdot \eta = \mathrm{id} : \flat \to \flat$$

plus some coherence conditions for tw

Some axioms for triangulated type theory I

Axiom (Interval $\mathbb{I}$)

There is a bounded distributive lattice $(\mathbb{I} : \mathrm{Set}, 0, 1, \vee, \wedge)$

Axiom (Reversal on $\mathbb{I}$)

There is an equivalence $\neg : \langle \mathrm{op} \mid \mathbb{I} \rangle \to \mathbb{I}$ *which swaps* 0 *for* 1 *and* $\wedge$ *for* $\vee$.

Axiom ($\mathbb{I}$ detects discreteness)

If $A :_\flat \mathcal{U}$ *then* $\langle \flat \mid A \rangle \to A$ *is an equivalence if and only if* $A \to (\mathbb{I} \to A)$ *is an equivalence.*

Axiom (Global points of $\mathbb{I}$)

The map $\mathbb{B} \to \mathbb{I}$ *is injective and induces an equivalence* $\mathbb{B} \simeq \langle \flat \mid I \rangle$.

Some axioms for triangulated type theory II

Axiom (Cubes separate)

$f :_\flat A \to B$ is an equivalence if and only if the following holds:

$$\Pi_{n :_\flat \mathbb{N}} \operatorname{isEquiv}\left(f_* : \langle \flat \mid \mathbb{2}^n \to A \rangle \to \langle \flat \mid \mathbb{2}^n \to B \rangle\right)$$

Compare/recall from Felix's and Ulrik's talks:

Axiom (Synthetic quasi-coherence (SQC) [Ble23])

Let $\mathbb{2} \to A$ be a finitely presented $\mathbb{2}$-algebra, i.e., $A \simeq \mathbb{2}[x_1, \ldots, x_n]/(r_1 = s_1, \ldots, r_n = s_n)$, then the evaluation map is an equivalence:

$$\mathrm{ev} \equiv \lambda a, f. f(a) : A \simeq (\hom_{\mathbb{2}}(A, \mathbb{2}) \to \mathbb{2})$$

Versions of the latter axiom appear in synthetic differential geometry (Kock–Lawvere axioms) [Koc77], synthetic algebraic geometry [CCH24], and synthetic domain theory [SY25]. For more context and a new categorical discussion see [Mye25].

Applications of SQC I

Lemma (Phoa's principle [Pho90] and [PS25])

$(\mathbb{I} \to \mathbb{I}) \simeq \Delta^2 \to \mathbb{I} \times \mathbb{I}$

A version of this also appears in ongoing work on synthetic Stone duality [CGM25], see also Hugo's upcoming talk and [Che+24]:

Lemma (Generalized Phoa's principle)

- $(\mathbb{I}^n \to \mathbb{I}) \simeq \mathbf{Pos}(\mathbb{B}^n, \mathbb{I})$
- $(\Delta^n \to \mathbb{I}) \simeq \mathbf{Pos}([0 \leq \ldots \leq n], \mathbb{I})$

Theorem

- $\mathbb{I}$ *is simplicial.*
- Δ^n *is a category.*

Applications of SQC II

Theorem

If $A :_\flat \mathcal{U}$ is discrete then A is simplicial.

After Gratzer, using $\flat \dashv \sharp$ one can prove that $\mathbb{N}$ is discrete.

Corollary

$\mathbb{B}$ and $\mathbb{N}$ are both discrete and simplicial, i.e., groupoids.

Warning: The theorem is *false* for general Rezk types, e.g. consider $\Delta^2 \amalg_{\Delta^1} \Delta^2$.

Towards the ∞-category of ∞-groupoids

- **Covariant families** are ∞-groupoids fibered over ∞-categories.
- They admit **transport**: $(-)_! : \prod_{a,b:X} (a \to_X b) \to A(a) \to A(b)$
- If X is Segal, then each fiber $A(a)$ is discrete.
- Can we take $\sum_{A:\mathcal{U}} \mathrm{isCov}(A)$?
- **No**: $\mathrm{isCov}(A)$ just means that A is discrete; doesn't see variance.
- Need a predicate that yields covariance **over all possible contexts**.
- **Solution: Amazing fibrations** due to M. Riley (2024): *A Type Theory with a Tiny Object* , arXiv:2403.01939; based on Licata–Orton–Pitts–Spitters '18 (which was used for similar purposes by Weaver–Licata '20)

Amazingly covariant families

- Consider $\mathrm{isCov}(A : \square \to \mathcal{U}) \simeq \prod_{x:A(0)} \mathrm{isContr}\left(\sum_{y:A(1)} (x \to_\alpha y)\right)$, where $\alpha : \mathrm{hom}_\square(0,1)$.
- This gives a predicate $\mathrm{isCov}_\square : \mathcal{U}^\square \to \mathrm{Prop}$.

Definition (Amazingly covariant types)

Let $A : \mathcal{U}$ be a type. It is *amazingly covariant* if and only if the following proposition is inhabited:

$$\mathrm{isaCov}(A) :\equiv \big(\mathrm{isCov}_\square(\lambda i. A^\eta(i))\big)_\square,$$

where A^η is the image of A under the unit $\eta_\mathcal{U} : \mathcal{U} \to (\mathcal{U}^\square)_\square$.

The ∞-category of ∞-groupoids

Define the universe of simplicial types and ∞-groupoids, resp.:

$$\mathcal{U}_{\boxslash} :\equiv \sum_{A:\mathcal{U}} \mathrm{isSimp}(A) \qquad \mathcal{S} :\equiv \sum_{A:\mathcal{U}_{\boxslash}} \mathrm{isaCov}(A)$$

Theorem (The ∞-category of ∞-groupoids [GWB24])

1. *$\mathcal{S}$ classifies amazingly covariant families in $\mathcal{U}_{\boxslash}$:*

$$\begin{tikzcd} E \simeq B \times_{\mathcal{S}} \mathcal{S}_* \arrow[r] \arrow[d, two heads, "\xi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{S}_* \arrow[d, two heads, "\pi"] \\ B \arrow[r, "\chi_\xi"] & \mathcal{S} \end{tikzcd}$$

2. *$\mathcal{S}$ is closed under Σ, identity types, and finite (co)limits.*
3. *$\mathcal{S}$ is* ***directed univalent****:*

$$\mathrm{arrtofun} : (\Delta^1 \to \mathcal{S}) \simeq \left(\sum_{A,B:\mathcal{S}} (A \to B) \right)$$

4. *$\mathcal{S}$ is Segal and Rezk, i.e., an ∞-category.*

Directed univalence

1. Since $\mathcal{S}$ classifies (amazingly) covariant families, there is a map

$$\mathrm{arrtofun} :\equiv \lambda F.(F\,0, F\,1, \alpha_!^F : F\,0 \to F\,1) : (\Delta^1 \to \mathcal{S}) \to \Big(\sum_{A,B:\mathcal{S}} (A \to B)\Big).$$

2. In the other direction, we consider the **mapping cone/directed glue type** (cf. cubical type theory and Weaver–Licata '20):

$$\mathrm{Gl} :\equiv A, B, f.\lambda i. \sum_{b:B} (i = 0) \to f^{-1}(b) : \Big(\sum_{A,B:\mathcal{S}} (A \to B)\Big) \to (\Delta^1 \to \mathcal{S})$$

3. Then argue that these are inverses up to homotopy.

Application: directed structure identity principle (DSIP)

Theorem (DSIP for pointed spaces)

Let $\mathcal{S}_* :\equiv \sum_{A:\mathcal{S}} A$. *Then for* $(A,a),(B,b) : \mathcal{S}_*$ *we have:*

$$\hom_{\mathcal{S}_*}((A,a),(B,b)) \simeq \sum_{f:A\to B} f(a) = b$$

Theorem (DSIP for monoids)

Consider the type (category!) of (set-)monoids

$$\mathrm{Monoid} :\equiv \sum_{A:\mathcal{S}_{\leq 0}} \sum_{\varepsilon:A} \sum_{\cdot:A\times A} \mathrm{isAssoc}(\cdot) \times \mathrm{isUnit}(\cdot,\varepsilon).$$

Then homomorphisms from $(A,\varepsilon_A,\cdot_A,\alpha_A,\mu_A)$ *to* $(B,\varepsilon_B,\cdot_B,\alpha_B,\mu_B)$ *correspond to set maps* $A\to B$ *compatible with multiplication and units.*

Towards synthetic higher algebra

We can internally define presheaf categories $\mathrm{PSh}(C) :\equiv \langle \mathrm{op}|C\rangle \to \mathcal{S}$.

Definition (∞-monoids)

The category Mon_∞ of ∞*-monoids* is the full subcategory[a] of $\mathrm{PSh}(\Delta)$ defined by the predicate

$$\varphi(X :_\flat \mathrm{PSh}(\Delta)) :\equiv \prod_{n:\mathrm{Nat}} \mathrm{isEquiv}(\langle X(\iota_k)_{k<n}\rangle : X(\Delta^n) \to X(\Delta^1)^n)$$

[a] need the codiscrete modality $\sharp$

This encodes the structure of a homotopy-coherent monoid. Multiplication is given through

$$\mu_X : X(\Delta^1) \simeq X(\Delta^1)^2 \to X(\Delta^1).$$

Definition (∞-groups)

The category Grp_∞ of ∞*-groups* is the full subcategory of Mon_∞ defined by the predicate

$$\varphi(X :_\flat \mathrm{Mon}_\infty) :\equiv \mathrm{isEquiv}(\lambda x, y.\langle x, \mu_X(x,y)\rangle : X(\Delta^1)^2 \to X(\Delta^1)^2)$$

Again using DSIP, these categories have the right types of morphisms.

Functorial Yoneda lemma and presheaf theory

Let $\Phi_C : \langle \mathrm{op} \mid C \rangle \times C \to \mathcal{S}$ be the unstraightening of the **twisted arrow fibration.** Then define the **Yoneda embedding** $\mathbf{y}_C := \lambda c.\Phi_C(-, c) : C \to \widehat{C} := \mathcal{S}^{\langle \mathrm{op} \mid C \rangle}$.

Theorem (Functorial Yoneda lemma)

There is a natural isomorphism $\Phi_C(\mathbf{y}(-), -) \cong \mathrm{eval} : \langle \mathrm{op} \mid C \rangle \times \widehat{C} \to \mathcal{S}$.

Theorem (Density)

If $X :_\flat \widehat{C}$, *then* $X \simeq \varinjlim_{\langle \mathrm{op} \mid \widetilde{X} \rangle} \mathbf{y} \circ \pi^{\mathrm{op}}$.

Theorem (Descent)

Let A *be a category and* $F :_\flat C \to \widehat{A}$, *then* $\widehat{A}/\varinjlim_{c:C} F(c) \simeq \varprojlim_{c:C} \widehat{A}/F(c)$.

Theorem (Cocompletion)

$\widehat{C}$ *is the free cocompletion of* C: $\mathbf{y}^* : (\widehat{C} \to_{\mathrm{cc}} E) \simeq (C \to E)$

Kan extensions, cofinal functors, and Quillen's Theorem A

Definition (Kan extensions)

Given $f :_\flat C \to D$ and a category E, the **left Kan extension** lan_f is the left adjoint to $f^* : E^D \to E^C$.

Theorem (Colimit formula)

Let E be cocomplete and $X :_\flat C \to E$, then $\mathrm{lan}_f X\, d \simeq \varinjlim(C \times_D D/d \to C \to E)$.

Definition (Cofinal functors)

A functor $f :_\flat C \to D$ is **right cofinal** if for every $X :_\flat D \to \mathcal{S}$ we have $\varinjlim_D X \simeq \varinjlim_C X \circ f$.

Proposition (Characterization of right cofinality)

A functor is right cofinal iff it is left orthogonal to all right fibrations.

Theorem (Quillen's Theorem A [GWB25])

A functor $f :_\flat C \to D$ is right cofinal if and only if $L_{\square}(C \times_D d/D) \simeq \mathbf{1}$ for each $d :_\flat D$.

Sifted colimits

Definition

A crisp ∞-category C is **sifted** if $\varinjlim_{C} : \mathcal{S}^{C} \to \mathcal{S}$ preserves finite products.

With Quillen's Theorem A we get:

Proposition

A crisp ∞-category C is sifted if and only if for all $n : \mathbb{N}$ the map $C^{!} : C \to C^{n}$ is right cofinal.

Theorem

If C has finite coproducts and sifted colimits then it is cocomplete.

Filtered colimits

Definition

An ∞-category is **finite** if it is generated by $\mathbf{0}$, $\mathbf{1}$, or ▯ under pushouts.

Definition

A crisp ∞-category C is **filtered** if $\varinjlim : \mathcal{S}^C \to \mathcal{S}$ preserves finite limits.

Definition

A crisp ∞-category C is **weakly filtered** if $C^! : \mathcal{C}^X \to \mathcal{C}$ is right cofinal for all finite ∞-categories X.

We can adapt [SW25] to prove:

Theorem

If C has finite and filtered colimits then it is cocomplete.

Spectra

Stable homotopy theory studies the limit behavior of spaces upon repeatedly suspending them. Spaces get replaced by *spectra* which correspond to symmetric monoidal ∞-groupoids and are central to higher algebra:

Definition (The ∞-category of spectra)

The ∞-category of **spectra** is defined as the (HoTT) limit: $\mathrm{Sp} :\equiv \varprojlim(\mathcal{S}_* \xleftarrow{\Omega} \mathcal{S}_* \xleftarrow{\Omega} \dots)$.

Proposition

Sp *is closed under finite limits and filtered colimits.*

Following [Cno25], using the cofinality of $\mathbb{N}$ we can prove:

Proposition

$\Omega : \mathrm{Sp} \to \mathrm{Sp}$ *is an equivalence.*

Proposition

Sp *is finitely cocomplete, and pushouts coincide with pullbacks.*

Corollary

Sp *is cocomplete.*

Ordinary homology theories and smash product

For a commutative ring R, consider the **Eilenberg–Mac Lane spectrum functor** $1 \mapsto \mathrm{H}R : \mathcal{S} \to \mathrm{Sp}$.

Theorem

The family of functors $H_i : \mathcal{S} \to \mathrm{Ab}$ defined by $H_i\, X :\equiv \pi_i\, \mathrm{H}(X; R)$ satisfies the Eilenberg–Steenrod axioms.

Via directed univalence we can define the smash product $- \wedge - : \mathcal{S}_* \times \mathcal{S}_* \to \mathcal{S}_*$, immediately recovering the results proven in Book HoTT such as associativity [Lju24]. Using directed univalence again, we can lift the following to a functor on spectra:

Definition

The **smash product** of spectra $X, Y : \mathrm{Sp}$ is given by

$$X \otimes Y :\equiv \varinjlim_{i,j:\mathbb{N}} \Omega^{i+j} \Sigma^\infty (X_i \wedge Y_j).$$

The ∞-category of ∞-categories

- Recently, we constructed the long desired **∞-category of ∞-categories** Cat synthetically.
- Just as $\mathcal{S}$ is the classifier for (amazingly) covariant fibrations, Cat should be the classifier for (amazingly) **cocartesian fibrations**, see [BW23].
- A direct attempt to define Cat in this way seems intractible. However, an approach via **locally cocartesian arrows** works. Over $\Box$ this amounts to:

$$\mathrm{isLCC} : \prod_{A:\mathcal{U}^{\Box}} \prod_{a:(i:\Box)\to A(i)} \mathrm{Prop}$$

$$\mathrm{isLCC}(A,a) :\equiv \prod_{b:(i:\Box)\to A(i)} \prod_{p:a0=b0} \mathrm{isContr}\left(\sum_{t:(i,j:\Delta^2)\to A(i)} t|_{\Lambda_0^2} = [a,b,p] \right)$$

- We then define Cat by:

$$\mathrm{Cat} :\equiv \sum_{A:\mathcal{U}_{\boxslash}} \mathrm{isRezk}(A) \times \mathrm{aIsInner}(A) \times \mathrm{aHasLCCLifts}(A) \times \mathrm{aLCCLiftsCompose}(A)$$

Classifying property and directed univalence

Theorem (Gratzer–W–Buchholtz '25)

1. Cat *classifies (amazingly) cocartesian families in* $\mathcal{U}_{\boxslash}$*:*

$$\begin{tikzcd} E \simeq B \times_{\mathrm{Cat}} \mathrm{Cat}_* \arrow[r] \arrow[d, two heads, "\xi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_* \arrow[d, two heads, "\pi"] \\ B \arrow[r, "\chi_\xi"] & \mathrm{Cat} \end{tikzcd}$$

2. Cat *is directed univalent:*

$$\mathsf{arrtofun} : (\Delta^1 \to \mathrm{Cat}) \simeq \left(\sum_{A,B:\mathrm{Cat}} (A \to B) \right)$$

3. Cat *is a category.*

Cocartesian directed gluing

Definition

Let $F_0, F_1 :_\flat X \to \mathcal{U}_{\boxslash}$ be cocartesian fibrations, and $\alpha :_\flat \prod_{x:X} F_0(x) \to F_1(x)$ be a cocartesian functor. The **directed gluing** is given by:

$$\mathrm{Gl}(F_0, F_1, \alpha) : \mathbb{I} \times X \to \mathcal{U}_{\boxslash}$$
$$\mathrm{Gl}(F_0, F_1, \alpha)(i, x) :\equiv \sum_{f : F_1(x)} (i = 0) \to \alpha(x)^{-1}(f)$$

Theorem

If X is a category and F_0, F_1, α are as above, then $\mathrm{Gl}(F_0, F_1, \alpha)$ is cocartesian.

Outlook

- more on Conduché fibrations and the $(\infty, 2)$-topos perspective, see [AM24]
- synthetic ∞-monads, ∞-operads, (symmetric) monoidal ∞-categories, ...
- $(\mathrm{Sp}, \otimes, \mathbb{S})$ as an s.m.c. (or the unit in presentable stable ∞-categories)
- internal higher topos theory
- metatheory of (higher?) type theories internally in type theory
- computational version and metatheory of modal sHoTT
- Connections to synthetic higher & differential geometry [Sch13; SS12; Shu18; Wel18; CCH24; MR23] and synthetic Stone duality & (light) condensed type theory ([BC24; Che+24; CGM25])
- more formalization
- ...

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