Modest sets and equilogical spaces as mono-fibrational cocompletions

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In memoriam: Thomas Streicher (1958-2025)

Triposes

Definition

A Set-tripos[1] is an **indexed poset**

$$\mathcal{P} : \mathsf{Set}^{\mathsf{op}} \to \mathsf{Pos}$$

such that:

- For all sets I, the poset $\mathcal{P}(I)$ is a **Heyting algebra**.
- For all functions $f : I \to J$, the **reindexing map** $f^* : \mathcal{P}(J) \to \mathcal{P}(I)$ is a **Heyting algebra morphism** and has left and right adjoints $\exists_f \dashv f^* \dashv \forall_f$ satisfying the **Beck-Chevalley condition**:

  (BCC) For all pullback squares $\begin{array}{ccc} A & \xrightarrow{h} & B \\ k\downarrow & \lrcorner & \downarrow g \\ C & \xrightarrow{f} & D \end{array}$ in Set, we have $g^* \circ \exists_f = \exists_h \circ k^*$ and $g^* \circ \forall_f = \forall_h \circ k^*$.
- There exists a **generic predicate**, i.e. a set Σ and a predicate $\mathrm{tr} \in \mathcal{P}(\Sigma)$ such that for all sets A and elements $\phi \in \mathcal{P}(A)$ there exists an $f : A \to \Sigma$ with $f^*(\mathrm{tr}) = \phi$.

Triposes were introduced as an auxiliary tool in the construction of **realizability toposes** from **partial combinatory algebras** (PCAs), notably Hyland's **effective topos**[2].

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[1] Hyland, Johnstone, and Pitts. "Tripos theory". In: *Math. Proc. Cambridge Philos. Soc.* (1980)

[2] Hyland. "The effective topos". In: *The L.E.J. Brouwer Centenary Symposium (Noordwijkerhout, 1981)*. 1982.

Realizability triposes

Definition

The **effective tripos** $\mathbf{eff} : \mathsf{Set}^{\mathsf{op}} \to \mathsf{Preord}$ is given by

$$\mathbf{eff}(I) = (P(\mathbb{N})^I, \leq)$$

where

$$(\phi : I \to P(\mathbb{N})) \leq (\psi : I \to P(\mathbb{N})) \quad \text{iff} \quad \exists (f : \mathbb{N} \xhookrightarrow{\text{part. rec.}} \mathbb{N})\, \forall (i \in I)\, \forall (n \in \phi(i))\, .\, f(n) \in \psi(i)$$

More generally:

Definition

Let $\mathcal{A}$ be a **partial combinatory algebra (PCA)**. The **realizability tripos** $\mathsf{rt}(\mathcal{A}) : \mathsf{Set}^{\mathsf{op}} \to \mathsf{Preord}$ is given by

$$\mathsf{rt}(I) = (P(\mathcal{A})^I, \leq)$$

where

$$(\phi : I \to P(\mathcal{A})) \leq (\psi : I \to P(\mathcal{A})) \quad \text{iff} \quad \exists (e \in \mathcal{A})\, \forall (i \in I)\, \forall (a \in \phi(i))\, .\, e \cdot a \in \psi(i)$$

Characterization of realizability triposes over PCAs

Theorem (F)[3]

A tripos $\mathcal{P} : \mathsf{Set}^{\mathsf{op}} \to \mathsf{Pos}$ is a realizability tripos over a PCA, iff :

(1) $\mathcal{P}$ has enough $\exists$-**prime predicates**.

(2) The full indexed sub-poset $\mathcal{A} = \mathsf{prim}(\mathcal{P}) \subseteq \mathcal{P}$ of $\exists$-prime predicates has finite meets.

(3) $\mathcal{A}$ has a **discrete** generic predicate.

(4) $\mathcal{A}$ is **shallow**, i.e. $\mathcal{A}(1) = 1$

Here:

- A predicate $\pi \in \mathcal{P}(I)$ is called $\exists$-**prime** if all its reindexings have the **left lifting property** (LLP) w.r.t. **cocartesian maps** in the total category $\int \mathcal{P}$.
- A predicate $\delta \in \mathcal{A}(I)$ is called **discrete**, if it has the **right lifting property** (RLP) w.r.t. **cartesian maps over surjections** in the total category $\int \mathcal{A}$ $(= \mathsf{PAsm}(\mathcal{A}))$.
- (1) means that $\mathcal{P}$ is a **cocompletion**, and (3) means that $\mathcal{A}$ is a **completion**.
- Thus, realizability triposes are **cocompletions of completions** (combined via a distributive law), which we'll analyze in this talk.

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[3] Frey. "A fibrational study of realizability toposes". PhD thesis. Paris 7 University, 2013
Frey. *Uniform Preorders and Partial Combinatory Algebras*. arxiv 2024, accepted in TAC

Fibrations vs indexed categories

Definition

A functor $p : \mathbb{E} \to \mathbb{B}$ is a **Grothendieck fibration**, if for all $E \in \mathbb{E}$, the functor $\mathbb{E}{\downarrow}E \to \mathbb{B}{\downarrow}p(E)$ is a **strict reflection**, i.e. it has a right adjoint section.

- For categories $\mathbb{C}$ in a fixed universe (i.e. 'small') we have a biequivalence

$$\mathsf{Fib}(\mathbb{C}) \simeq [\mathbb{C}^{\mathsf{op}}, \mathsf{Cat}]$$

  where $\mathsf{Fib}(\mathbb{C})$ is the 2-category of Grothendieck fibrations over $\mathbb{C}$ with small domain, and $[\mathbb{C}^{\mathsf{op}}, \mathsf{Cat}]$ the 2-category of pseudofunctors, pseudo-natural transformations, and modifications.
- This restricts to a biequivalence of locally ordered categories

$$\mathsf{Fib}_{\mathsf{ff}}(\mathbb{C}) \simeq [\mathbb{C}^{\mathsf{op}}, \mathsf{Pos}]$$

  between (amnestic) **faithful fibrations** and **indexed posets**.
- In the following we use faithful fibrations as analogues of indexed posets over Set, but there's a **size mismatch**: in general the fibers won't be small (but they will if the fibration has a generic predicate, such as a tripos).
- As a basis for our analysis, we introduce a more basic locally ordered category: FIFib is the category of **faithful isofibrations** (a.k.a. **concrete categories**) over Set.
- Notation: instead of $(U : \mathbb{C} \to \mathsf{Set}) \in \mathsf{FIFib}$ write $\mathbb{C} \in \mathsf{FIFib}$ and always write U for the functor.

Four monads

We consider four monads on FIFib

- T_{inj} : FIFib $\to$ FIFib freely adds **cartesian** lifts along **injections**.
- T_{surj} : FIFib $\to$ FIFib freely adds **cartesian** lifts along **surjections**.
- S_{inj} : FIFib $\to$ FIFib freely adds **cocartesian** lifts along **injections**.
- S_{surj} : FIFib $\to$ FIFib freely adds **cocartesian** lifts along **surjections**.

All these are given by similar constructions. For example, for $\mathbb{C} \in$ FIFib, the category $T_{\mathrm{inj}}\mathbb{C}$ has pairs $(C \in \mathbb{C}, m : S \rightarrowtail UC)$ as objects, and morphisms $(C, m : X \rightarrowtail UC) \to (D, n : Y \rightarrowtail UD)$ are given by by functions $f : X \to Y$ such that there exists a $g : C \to D$ with $Ug \circ m = n \circ f$.

$$\begin{tikzcd} X \arrow[r, tail, "m"] \arrow[d, "f"] & UC \arrow[d, "Ug"] & C \arrow[d, dashed, "g"] \\ Y \arrow[r, tail, "n"] & UD & D \end{tikzcd}$$

The 'underlying set' functor is given by $U(C, m : X \rightarrowtail UC) = X$

Remarks:

- We only require that g 'exists' since contrary to Quentin yesterday, we're freely generating **faithful** fibrations.
- For T_{surj} and S_{inj} this doesn't make a difference by cancellation properties.
- S_{inj} and S_{surj} are **lax idempotent**, and S_{inj} and S_{surj} are **colax idempotent**.

Distributive laws

Definition

Given monads $S, T : \mathbb{C} \to \mathbb{C}$ on a category $\mathbb{C}$, a **distributive law** is a natural transformation $\delta : TS \to ST$ satisfying certain axioms.

Proposition (Beck, ?)

TFAE:

- distributive laws $\delta : TS \to ST$
- monad structures on ST satisfying certain conditions
- 'liftings' of S to the category $\mathbb{C}^T$ of T-algebras
$$\begin{tikzcd} \mathbb{C}^T \arrow[r, dashed] \arrow[d, "U"] & \mathbb{C}^T \arrow[d, "U"] \\ \mathbb{C} \arrow[r, "S"] & \mathbb{C} \end{tikzcd}$$
- 'extensions' of T to the Kleisli category $\mathbb{C}_S$ of S
$$\begin{tikzcd} \mathbb{C} \arrow[r, "T"'] \arrow[d, "F"] & \mathbb{C} \arrow[d, "F"] \\ \mathbb{C}_S \arrow[r, dashed] & \mathbb{C}_S \end{tikzcd}$$

Claim: In general, there may be many distributive laws between two monads S, T. However, if T is 'property-like' (e.g. lax idempotent or colax idempotent), then there is at most one, and it exists iff S maps T-algebras to T-algebras.

Monadic lifting

Given a distributive law $\delta : TS \to ST$ we get a square of categories and forgetful functors where three sides (and the diagonal) are monadic.

$$\begin{tikzcd}
\mathbb{C}^{ST} \arrow[r, shift left=2] \arrow[d, shift left=2] \arrow[r, phantom, "\top"] \arrow[d, phantom, "\dashv"] & \mathbb{C}^{T} \arrow[l, shift left=2] \arrow[d, shift left=2] \arrow[d, phantom, "\dashv"] \\
\mathbb{C}^{S} \arrow[u, dashed, shift left=2] \arrow[r, shift left=2] \arrow[r, phantom, "\top"] & \mathbb{C} \arrow[l, shift left=2] \arrow[u, shift left=2]
\end{tikzcd}$$

By adjoint lifting and, adjoint on the left exists whenever $\mathbb{C}^{ST}$ has reflexive coequalizers, which is very often the case. The adjunction is then automatically monadicmonadic cancellation[4].
Examples:

$$\begin{tikzcd}
\mathsf{Sup} \arrow[r, shift left=2] \arrow[d, shift left=2] \arrow[r, phantom, "\top"] \arrow[d, phantom, "\dashv"] & \mathsf{jSLat} \arrow[l, shift left=2] \arrow[d, shift left=2] \arrow[d, phantom, "\dashv"] \\
\mathsf{DCPO} \arrow[u, dashed, shift left=2] \arrow[r, shift left=2] \arrow[r, phantom, "\top"] & \mathsf{Pos} \arrow[l, shift left=2] \arrow[u, shift left=2]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Frm} \arrow[r, shift left=2] \arrow[d, shift left=2] \arrow[r, phantom, "\top"] \arrow[d, phantom, "\dashv"] & \mathsf{mSLat} \arrow[l, shift left=2] \arrow[d, shift left=2] \arrow[d, phantom, "\dashv"] \\
\mathsf{Sup} \arrow[u, dashed, shift left=2] \arrow[r, shift left=2] \arrow[r, phantom, "\top"] & \mathsf{Pos} \arrow[l, shift left=2] \arrow[u, shift left=2]
\end{tikzcd}$$

Empirical observation: If the RHS adjunction is (co)lax idempotent then the LHS is as well, but typically not **mnemetic** (cf. Quentin's talk). Source of interesting LNL adjunctions.

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[4]See this Zulip discussion, thanks to Tom Hirschowitz and Mathieu Anel.

Many distributive laws

I claim that the monads $T_{\text{inj}}, T_{\text{surj}}, S_{\text{inj}}, S_{\text{surj}}$, admit distributive laws **for any distinct pair in both directions**. We're interested specifically in

- $T_{\text{inj}} \circ T_{\text{surj}} \to T_{\text{surj}} \circ T_{\text{inj}} =: T_{\text{all}}$
  (free faithful fibration monad, arising from epi-mono factorization system)
- $S_{\text{surj}} \circ S_{\text{inj}} \to S_{\text{inj}} \circ S_{\text{surj}} =: S_{\text{all}}$
  (free faithful opfibration monad, arising from epi-mono factorization system)
- $T_j \circ S_i \to S_i \circ T_j =: B_i^j$ for $i, j \in \{\text{surj}, \text{inj}, \text{all}\}$, (free faithful BC-bifibrations, arising from pullbacks)

Characterization of free faithful fibrations

Proposition

- A faithful mono-fibration is free over a faithful iso-fibration iff it has enough **injective objects** (RLP w.r.t. cartesian maps over injections)
- A faithful fibration is free over a faithful mono-fibration iff it has enough **discrete objects** (RLP w.r.t. cartesian maps over surjections)
- A faithful fibration is free over a faithful iso-fibration iff it has enough **discrete injective objects** (RLP w.r.t. all cartesian maps)

Grid of monadic functors

Together, we get the following grid of locally ordered categories of faithful BCC-bifibrations over Set, and monadic (co)lax idempotent adjunctions between them.

$$
\begin{tikzcd}
\mathsf{FFib}^{\mathsf{all}}_{\mathsf{all}} \arrow[r, shift left=1.5ex] \arrow[r, phantom, "\top"] & \mathsf{FFib}^{\mathsf{all}}_{\mathsf{surj}} \arrow[l, shift left=1.5ex] \arrow[r, shift left=1.5ex] \arrow[r, phantom, "\top"] & \mathsf{FFib}^{\mathsf{all}}_{\mathsf{iso}} \arrow[l, shift left=1.5ex] \\
\mathsf{FFib}^{\mathsf{inj}}_{\mathsf{all}} \arrow[u, shift left=1.5ex] \arrow[u, phantom, "\dashv"] \arrow[r, shift left=1.5ex] \arrow[r, phantom, "\top"] & \mathsf{FFib}^{\mathsf{inj}}_{\mathsf{surj}} \arrow[u, shift left=1.5ex] \arrow[u, phantom, "\dashv"] \arrow[l, shift left=1.5ex] \arrow[r, shift left=1.5ex] \arrow[r, phantom, "\top"] & \mathsf{FFib}^{\mathsf{inj}}_{\mathsf{iso}} \arrow[u, shift left=1.5ex] \arrow[u, phantom, "\dashv"] \arrow[l, shift left=1.5ex] \\
\mathsf{FFib}^{\mathsf{iso}}_{\mathsf{all}} \arrow[u, shift left=1.5ex] \arrow[u, phantom, "\dashv"] \arrow[r, shift left=1.5ex] \arrow[r, phantom, "\top"] & \mathsf{FFib}^{\mathsf{iso}}_{\mathsf{surj}} \arrow[u, shift left=1.5ex] \arrow[u, phantom, "\dashv"] \arrow[l, shift left=1.5ex] \arrow[r, shift left=1.5ex] \arrow[r, phantom, "\top"] & \mathsf{FFib}^{\mathsf{iso}}_{\mathsf{iso}} \arrow[u, shift left=1.5ex] \arrow[u, phantom, "\dashv"] \arrow[l, shift left=1.5ex]
\end{tikzcd}
$$

The superscript i in $\mathsf{FFib}^{\mathsf{i}}_{\mathsf{j}}$ denotes along which functions there are cartesian liftings, and the subscript j corresponds to cocartesian liftings.

E.g., $\mathsf{FFib}^{\mathsf{inj}}_{\mathsf{surj}}$ is the locally ordered category of faithful functors $U : \mathbb{C} \to \mathsf{Set}$ admitting cartesian liftings along injections, and cocartesian liftings along surjections, subject to BCC for all suitable squares.

Assemblies

Realizability triposes over a pca $\mathcal{A}$ are freely generated by the category $\mathsf{MPAsm}(\mathcal{A})$ of **modest partitioned assemblies** in $\mathsf{FFib}^{\mathsf{inj}}_{\mathsf{iso}}$. Many of the intermediate 'partial' completions are also known categories:

$\int\mathsf{rt}(\mathcal{A})$	$\mathsf{Asm}(\mathcal{A})$	$\mathsf{PAsm}(\mathcal{A})$
	$\mathsf{Mod}(\mathcal{A})$	$\mathsf{MPAsm}(\mathcal{A})$
		$[\mathsf{Comp}(\mathcal{A})]$

The grid cells correspond to the positions in the diagram above.

The claim is that all the stated categories are (co)completions of MPasm($\mathcal{A}$) along the suitable left adjoints.

- $\mathsf{Asm}(\mathcal{A})$ is the full subcategory of the total category $\int\mathsf{rt}(\mathcal{A})$ of the tripos on predicates $\phi : I \to P(\mathcal{A})$ which are *pointwise nonempty*
- $\mathsf{PAsm}(\mathcal{A})$ is the full subcategory on predicates which are pointwise singletons
- $\mathsf{Mod}(\mathcal{A})$ is the full subcategory on predicates whose fibers are pairwise disjoint
- $\mathsf{MPAsm}(\mathcal{A}) = \mathsf{PAsm}(\mathcal{A}) \cap \mathsf{Mod}(\mathcal{A})$
- If the PCA $\mathcal{A}$ is **total**, we can even fill the bottom row: $\mathsf{Comp}(\mathcal{A})$ is the full subcategory of $\mathsf{MPAsm}(\mathcal{A})$ on retracts of $(\mathcal{A}, \mathrm{id})$.

Notably not in the picture: the realizability topos $\mathsf{RT}(\mathcal{A})$. It's not a concrete category! However, the middle and right columns embed fully faithfully into it.

Equilogical Spaces

Scott's category Equ of **equilogical spaces**[5] fits into a similar grid:

	$\mathsf{Top}_{\mathsf{reg/lex}}$	Top
	Equ	Top_{T_0}
		ContLat

Here, ContLat is the full subcategory of Top on continuous lattices with the Scott topology.
Relevant obervations:

- Top $\to$ Set is a faithful fibration, $\mathsf{Top}_{\mathsf{T}_0}$ $\to$ Set is a faithful mono-fibration.
- T_0-spaces have the r.l.p. w.r.t. cartesian maps over surjections, and every space is a cartesian lifting of a T_0 space along a surjection.
- Continuous lattices (with Scott topology) are injective w.r.t. subspace inclusions of T_0 spaces, and every T_0 space embeds into a continuous lattice (even into an algebraic lattice).
- Claim: going from the top right to the top middle grid cell is always a reg/lex completion.
- Observation: Equ is locally cartesian closed just like Mod, as observed by Rosolini[6].

[5] Bauer, Birkedal, and Scott. "Equilogical spaces". In: *Theoretical Computer Science* (2004).
[6] Rosolini. "The category of equilogical spaces and the effective topos as homotopical quotients". In: *Journal of Homotopy and Related Structures* (2016).

Posets

And another variation:

		Preord
	?	Pos
		CompLat

Here, CompLat is the full subcategory of Pos on complete lattices.
Relevant facts:

- Preord $\to$ Set is a fibration
- Posets have the r.l.p. w.r.t. surjective cartesian maps between preorders
- complete lattices are injective w.r.t. embeddings of posets, and every posets embeds into a complete lattice.

Claim: In the middle (?) we get an locally cartesian closed category again, since CompLat is CCC.

Thank you for your attention!

Dialectica?

Gödel's Dialectica interpretation has been analyzed in terms of distributive laws between $\exists$ and $\forall$ (or Σ and Π)[7][8][9].

Not clear how this relates to what is discussed here.

[7] Hofstra. "The Dialectica monad and its cousins". In: *Models, logics, and higher-dimensional categories: A tribute to the work of Mihály Makkai. Proceedings of a conference, CRM, Montréal, Canada, June 18–20, 2009*. Providence, RI: American Mathematical Society (AMS), 2011.

[8] Trotta, Spadetto, and Paiva. "Dialectica logical principles". In: *Logical foundations of computer science*. Springer, Cham, 2022.

[9] Trotta, Weinberger, and Paiva. "Skolem, Gödel, and Hilbert fibrations". In: *arXiv preprint arXiv:2407.15765* (2024).