Bar Induction and Preservation of Cardinals

from a joint work with Laura Fontanella

Jacopo Furlan

LIPN – Université Sorbonne Paris Nord
LACL – Université Paris Est Cretèil

September 10, 2025

Forcing allows for producing models of set theory, given a fixed model M—the *ground model*—and a poset $\mathbb{P} = (P, \leq, \mathbf{1}) \in M$.

This technique yields the least ZF-model containing M and a *generic* filter G, which is a filter of $\mathbb{P}$. The so-called *generic extension* is denoted by $M[G]$.

A poset $\mathbb{P}$ is κ-closed for some cardinal κ if for any $\gamma < \kappa$ and for any descending sequence $(p_\xi)_{\xi<\gamma}$ of elements in P, there exists $q \in P$ such that $q \leq p_\xi$ for any $\xi < \gamma$.

PROPOSITION

Given a κ-closed poset $\mathbb{P}$ in a fixed model M, the generic extension $M[G]$ satisfies that for any $\mu \leq \kappa$

$$\text{if } M \models \mu \textit{ is a cardinal} \text{ then } M[G] \models \check{\mu} \textit{ is a cardinal}$$

or «cardinals until κ are preserved».

In the context of complete boolean algebras, the analogue to κ-closure for a boolean algebra $\mathbb{B} = (B, \mathbf{0}, \mathbf{1}, \wedge, \neg)$ states that $B \setminus \{\mathbf{0}\}$ contains a dense κ-closed subset.

LEMMA

A complete boolean algebra $\mathbb{B}$ is κ-closed if and only if for any $\gamma < \kappa$ and for any sequence $(b_\xi)_{\xi<\gamma}$ of elements of $B \setminus \{\mathbf{0}\}$, if for any $\alpha < \gamma$, $\bigwedge_{\xi<\alpha} b_\xi \neq \mathbf{0}$, then $\bigwedge_{\xi<\gamma} b_\xi \neq \mathbf{0}$.

Let's try to do the same for realizability algebras!

Following
Berardi, Bezem, Coquand, On the computational content of the axiom of choice (1998)

Krivine, Bar recursion in classical realisability (2016)

To find an analogue of κ-closure for realizability algebras, it is necessary to dispose of terms that represent ordinals below κ.

We then consider a realizability algebra $\mathcal{A} = (\Lambda_{\mathsf{c}}, \Pi, \succ, \perp\!\!\!\perp)$ which contains for any $\alpha < \kappa$, a term $\underline{\alpha}$ such that $(\underline{s})\underline{\alpha} \succ\!\!\succ \underline{\alpha + 1}$; furthermore, we assume the existence of a term χ such that

$$\chi \underline{\alpha}\, \underline{\beta} t u \succ\!\!\succ \begin{cases} t & \text{if } \alpha < \beta; \\ u & \text{otherwise} \end{cases}$$

for any $\alpha, \beta < \kappa$, $t, u \in \Lambda_{\mathsf{c}}$.

In addition to that, we also suppose the existence of a sequence $(u_\xi)_{\xi<\kappa}$ such that, for any $\alpha < \gamma < \kappa$, $(\underline{\gamma})\underline{\alpha} \succ\!\!\succ u_\alpha$.

We will say that a term t *accumulates* a sequence $(u_\xi)_{\xi<\gamma}$ whenever $(t)\underline{\alpha} \succ\!\!\succ u_\alpha$ for any $\alpha < \gamma$.

A realizability algebra is κ-closed for some cardinal κ if for any $\gamma < \kappa$ and for every sequence $(u_\xi)_{\xi<\gamma} \subseteq \Lambda_{\mathsf{c}}$

- there is a term t that accumulates $(u_\xi)_{\xi<\gamma}$;
- for any term v, if for any $\delta < \gamma$ there exists an accumulator w of the sequence $(u_\xi)_{\xi<\delta}$ such that $v\{x := w\} \not\Vdash \perp$, then for any accumulator t of the sequence $(u_\xi)_{\xi<\gamma}$, $v\{x := t\} \not\Vdash \perp$.

That is to say, "any $<\kappa$-sequence of terms can be recollected into a term t which preserves the compatibility of the sequence".

This property corresponds to κ-closure in forcing!

From a boolean algebra $\mathbb{B}$, we can build a realizability algebra $\mathcal{A}_{\mathbb{B}}$ that generates the same model.

Proposition

For any cardinal κ, $\mathbb{B}$ is κ-closed if and only if $\mathcal{A}_{\mathbb{B}}$ is κ-closed.

For the translation boolean algebras/realizability algebras see
Matthews, A guide to Krivine realizability for set theory (2023)

A generalized version of the result from Krivine (and Berardi Bezem Coquand) about the axiom of choice can be obtained.

THEOREM

Given a κ-closed realizability algebra $\mathcal{A}$, the realizability model N satisfies $\mathrm{AC}_{<\hat{\kappa}}$.

Furthermore, Krivine showed that under these circumstances N satisfies the continuum hypothesis when M does. We obtain the same result in the transfinite case.

Theorem

Given a κ-closed realizability algebra $\mathcal{A}$, if $M \models \mathrm{GCH}$, then $N \models \mathrm{CH}_{<\hat{\kappa}}$.

The technique already used by Berardi Bezem and Coquand for exhibiting a realizer of AC_ω can be reemployed to show that cardinals until κ are preserved.

The main tool for the proof is the bar induction operator, which is represented in our formalism by the term

$$\lambda gu.(Y)\lambda hif.(u)((\chi)if)(g)\lambda z.(h(\underline{s})i)(\chi)ifz$$

or **B** in the following.

Bar induction operator dates back to Spector's work on consistency of analysis and has been elaborated over forty years, until the famous article of Berger and Oliva.

This operator was originally defined as

$$\Phi(G,U,F,s) = \begin{cases} F(s) & \text{if } U(s@0) < len(s), \\ G(s, \lambda x.\Phi(G,U,F,s*x)) & \text{otherwise;} \end{cases}$$

where F, U, G are (continous) functionals and s is a finite sequence.

Spector, Provably recursive functionals of analysis (1962)

Berger, Oliva, Modified bar recursion and classical dependent choice (2001)

$$\Phi(G,U,F,s) = \begin{cases} F(s) & \text{if } U(s@0) < len(s), \\ G(s, \lambda x.\Phi(G,U,F,s*x)) & \text{otherwise;} \end{cases}$$

The halting condition $U : (\mathbb{N} \mapsto \mathbb{N}) \mapsto \mathbb{N}$ should be though of as an infinite branching tree of finite height, while $F : \mathbb{N} \mapsto \mathbb{N}$ and $G : \mathbb{N} \times (\mathbb{N} \mapsto \mathbb{N}) \mapsto \mathbb{N}$ are the operations to apply respectively in the base case (on the leaves) and in the inductive step.

The term Φ is then a third-order functional that stops when the prolongation of the initial input finally meets a leaf, or the *bar* U. The proof of termination crucially uses the Brouwerinan bar recursion principle.

The same kind of reasoning can be employed to prove the preservation of cardinals below κ in N.

Theorem

Given a κ-closed realizability algebra $\mathcal{A}$, there exists a term such that for any cardinal $\mu \leq \kappa$ and for any ordinal $\gamma < \mu$,

$$\forall f (Tot(f, \hat{\gamma}, \hat{\mu}) \wedge \forall g^{\hat{\gamma}} (Fun(g) \wedge (\mathtt{op}(x, g(x)) \,\varepsilon\, f) \to Surj(g)) \to \bot)$$

"PROOF"

For sake of readability, the reductum of $\mathbf{B}GUt\underline{\alpha}$ is rewritten as

$$(U)(\mathbf{Restr}[t,\underline{\alpha}])(G)\mathbf{Ind}[G,U,t,\underline{\alpha}].$$

Using this formalism, the stopping condition will be represented by U, while G makes the inductive operation. Let $\mu \leq \kappa$ be a cardinal, and consider an ordinal $\gamma < \mu$.

Fix a name $f \in M^{\mathcal{K}}$ and terms $G, L \in \Lambda_{\mathsf{c}}$ such that $G \in ||Tot(f, \hat{\gamma}, \hat{\mu})||^{\perp\!\!\!\perp}$ and $L \in ||F(f, \hat{\gamma}, \hat{\mu})||^{\perp\!\!\!\perp}$, where

$$F[f, \hat{\gamma}, \hat{\mu}] \equiv (\forall g(Fun(g), \neg Surj(g, \hat{\gamma}, \hat{\mu}), \forall x^{\hat{\gamma}} \mathsf{op}(x, \tilde{g}(x)) \,\varepsilon\, f) \to \bot).$$

Let $\theta_1 \Vdash Fun(\tilde{f})$ and θ_2 such that if g is not surjective, then $\theta_2 \Vdash \neg Surj(\tilde{g})$. Set $U = L\theta_1\theta_2$ to obtain

$$\lambda x.Ux \in ||\forall g(\forall x^{\hat{\gamma}}(\mathsf{op}(x, \tilde{g}(x)) \,\varepsilon\, f) \to \bot)||^{\perp\!\!\!\perp}.$$

Define $H = \mathbf{B}GU$ so that for every $\alpha < |\mathcal{K}|$ and every term t, we have

$$H\underline{\alpha}t \succ (U)(\mathbf{Restr}[t, \underline{\alpha}])(G)\mathbf{Ind}[G, U, t, \underline{\alpha}].$$

We show by contradiction that $H\underline{0}\,\underline{0} \in ||\bot||^{\perp\!\!\!\perp}$, from which we deduce that $\lambda xy.(\mathbf{B}(y\theta_1\theta_2))x\underline{0}\,\underline{0}$ realizes the formula

$$\forall f(Tot(f, \hat{\gamma}, \hat{\mu}), F(f, \hat{\gamma}, \hat{\mu}) \to \bot).$$

Set $t_0 = \underline{0}$ and suppose by contradiction that $H\underline{0}t_0 \not\Vdash \perp$.

We define by induction two sequences of terms $(t_\alpha)_{\alpha<\gamma}$ and $(u_\alpha)_{\alpha<\gamma}$ and a sequence of ordinals $(\mu_\alpha)_{\alpha<\gamma}$ such that for any $\alpha < \gamma$, the following conditions hold:

1. t_α accumulates $(u_\xi)_{\xi<\alpha}$;
2. $H\underline{\alpha}t_\alpha \not\Vdash \perp$;
3. $u_\alpha \in ||\mathtt{op}(\hat{\alpha}, \widehat{\mu_\alpha})\varepsilon f||^{\perp\!\!\!\perp}$.

The machinery of bar induction is employed in the induction step of the definition of $(t_\xi, u_\xi, \mu_\xi)_{\xi<\gamma}$.

Suppose to have defined $(t_\xi, u_\xi, \mu_\xi)_{\xi<\alpha}$ for some $\alpha < \gamma$.
$\mathbf{Ind}[G, U, t_\alpha, \underline{\alpha}] \notin ||\forall x^{\hat{\mu}} \mathtt{op}(x, \hat{\alpha}) \not\varepsilon f||^{\perp\!\!\!\perp}$, otherwise $(G)\mathbf{Ind}[G, U, t_\alpha, \underline{\alpha}] \in ||\perp||^{\perp\!\!\!\perp}$ and

$$\mathbf{Restr}[t_\alpha, \underline{\alpha}](G)\mathbf{Ind}[G, U, t_\alpha, \underline{\alpha}] \in ||\forall x^{\hat{\mu}} \mathtt{op}(x, \tilde{g}(x)) \varepsilon f||^{\perp\!\!\!\perp}$$

where g is the function $g(\xi) = \mu_\xi$ for $\xi < \alpha$.

By definition of U, we would have
$\mathbf{Restr}[t_\alpha, \underline{\alpha}](G)\mathbf{Ind}[G, U, t_\alpha, \underline{\alpha}] \in ||\bot||^{\perp\!\!\!\perp}$, against the hypothesis.

$\mathbf{Ind}[G, U, t_\alpha, \underline{\alpha}] \notin ||\forall x^{\hat{\mu}} \mathsf{op}(x, \hat{\alpha}) \not\varepsilon f||^{\perp\!\!\!\perp}$ means that there exists u, δ such that $(\mathbf{Ind}[G, U, t_\alpha, \underline{\alpha}])u \notin ||\bot||^{\perp\!\!\!\perp}$ and

$$u \in ||\mathsf{op}(\hat{\alpha}, \hat{\delta})\, \varepsilon\, f||^{\perp\!\!\!\perp}.$$

Setting $t_{\alpha+1} = (\chi)\underline{\alpha}t_\alpha u$, $u_\alpha = u$ and $\mu_\alpha = \delta$ concludes the step.

This allows to find the sought sequence of terms (N.B.: the limit case is as interesting as technical, so omitted).

The accumulator of $(u_\xi)_{\xi<\gamma}$ satisfies $||\forall x^{\hat{\gamma}}\,\mathtt{op}(x, \tilde{g}(x))\,\varepsilon\, f||^{\perp\!\!\!\perp}$, hence $(U)v \in ||\bot||^{\perp\!\!\!\perp}$.

Since any accumulator should preserve compatibility, by (the contrapositive of) κ-closure there exists $\alpha < \gamma$ and an accumulator w of $(u_\xi)_{\xi<\alpha}$ such that $(U)w \in ||\bot||^{\perp\!\!\!\perp}$, against κ-closure.

It is possible to give an exact characterization of cardinal preservation in forcing.

Proposition

Given a poset $\mathbb{P}$ in a fixed model M, the generic extension $M[G]$ satisfies that for any cardinal $\mu \leq \kappa$

$$\text{if } \; M \models \mu \textit{ is a cardinal} \; \text{ then } \; M[G] \models \check{\mu} \textit{ is a cardinal}$$

if and only if $\mathbb{P}$ is κ-distributive.

Can we generalize this property to realizability algebras?

Thank you for the attention