

Towards the effective 2-topos

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joint work with Steve Awodey

with thanks to M. Anel, R. Barton, J. Frey, and P. Rosolini

Synthetic mathematics, logic-affine computations and efficient proof systems

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Introduction

Useful features of the effective topos $\mathcal{E}ff$:

- ▶ It is a(n elementary) topos, in particular locally cartesian closed.
- ▶ All functions on natural numbers are Turing-computable (Church Thesis).
- ▶ It contains a small subcategory which is complete but not a preorder (modest sets).

So $\mathcal{E}ff$ allows to construct realisability models of extensional type theories with an impredicative universe of types.

Hyland. The effective topos. 1982

Hyland. A small complete category. 1988

Hyland, Robinson, Rosolini. The discrete objects in the effective topos. 1990

Introduction

Current line of research: to identify higher versions of $\mathcal{E}ff$ that

- ▶ interpret intensional type theories with a univalent impredicative universe,
- ▶ and provide examples of elementary higher toposes.

Desiderata for every higher-dimensional effective topos:

it should contain $\mathcal{E}ff$ as the full subcategory of 0-types (=h-sets).

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Desiderata for every higher-dimensional effective topos:

it should contain $\mathcal{E}ff$ as the full subcategory of o-types (=h-sets).

Straightforward options (taking simplicial or cubical objects in $\mathcal{E}ff$ or $\mathcal{A}sm$) add o-types: $\mathcal{E}ff$ is already a completion (under quotients of equivalence relations), so taking $\mathcal{E}ff^{\Delta^{op}}$ adds those colimits again at the level of o-types.

An eq. relation $E \rightrightarrows X$ in $\mathcal{E}ff$ has two distinct quotients in $\mathcal{E}ff^{\Delta^{op}}$: the (discrete simplicial set on the) quotient X/E and (the nerve of) $E \rightrightarrows X$.

Today: a 2-dimensional proposal, using groupoids.

Awodey, E. Toward the effective 2-topos. arXiv:2503.24279

The effective topos as a colimit completion

- $\mathcal{E}ff$ is the exact completion of the regular category of assemblies $\mathcal{A}sm$:

$$\mathcal{E}ff = (\mathcal{A}sm)_{\text{ex/reg}}$$

Carboni, Freyd, Scedrov. A Categorical approach to Realizability and Polymorphic Types. 1988

- since $\mathcal{A}sm$ is the regular completion of the lex category of partitioned assemblies $\mathcal{PA}sm$, we also have:

$$\mathcal{E}ff = (\mathcal{A}sm)_{\text{ex/reg}} = (\mathcal{PA}sm)_{\text{ex/lex}}$$

Robinson, Rosolini. Colimit completions and the effective topos. 1990

Carboni. Some free constructions in realizability and proof theory. 1995

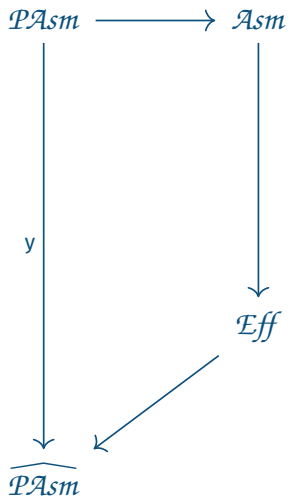
As such, $\mathcal{E}ff$ embeds into presheaves on $\mathcal{PA}sm$,
and can be characterised as a subcategory of $\widehat{\mathcal{PA}sm}$.

Lack. A note on the exact completion of a regular category, and its infinitary generalizations. 1999

Hu, Tholen. A note on free regular and exact completions and their infinitary generalizations. 1996

$$\begin{array}{ccc} \mathcal{PA}sm & \xrightarrow{y} & \widehat{\mathcal{PA}sm} \\ \downarrow & & \uparrow \\ \mathcal{A}sm & \longrightarrow & \mathcal{E}ff \end{array}$$

The effective topos as a colimit completion



$$yQ \twoheadrightarrow A \rightharpoonup yP$$

$$B \rightrightarrows A \twoheadrightarrow X$$

The effective topos as a colimit completion

$$\begin{array}{ccc}
 \mathcal{P}\mathcal{A}sm & \longrightarrow & \mathcal{A}sm \\
 \downarrow y & & \downarrow \\
 & & \mathcal{E}ff \\
 & & \downarrow \\
 \widehat{\mathcal{P}\mathcal{A}sm} & \longleftarrow & Cpt(\widehat{\mathcal{P}\mathcal{A}sm})
 \end{array}$$

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Outline

Give:

1. Definition of coherent groupoid
2. Quillen model structure on $\mathbf{Gpd}(\widehat{\mathcal{P}\mathcal{A}sm})$

such that:

$$\begin{array}{ccc} \mathcal{E}ff & \hookrightarrow & \mathbf{Gpd}_o(\widehat{\mathcal{P}\mathcal{A}sm}) \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{CohGpd}(\widehat{\mathcal{P}\mathcal{A}sm}) & \hookrightarrow & \mathbf{Gpd}(\widehat{\mathcal{P}\mathcal{A}sm}) \end{array}$$

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$$\begin{array}{ccc} \mathcal{E}ff & \hookrightarrow & \mathbf{Gpd}_o(\widehat{\mathcal{P}\mathcal{A}sm}) \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{Eff}_2 := \mathbf{CohGpd}(\widehat{\mathcal{P}\mathcal{A}sm}) & \hookrightarrow & \mathbf{Gpd}(\widehat{\mathcal{P}\mathcal{A}sm}) \end{array}$$

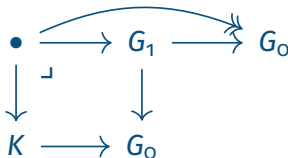
Compact groupoids

Say that a **compact discrete** groupoid is a discrete groupoid $\underline{K}$ on a compact presheaf K .

A groupoid $\mathbb{G}$ is **pseudo-compact** if there is

$$\underline{K} \rightarrow \mathbb{G}$$

essentially surjective on objects from a compact discrete groupoid $\underline{K}$, that is:



NB: A discrete groupoid $\underline{X}$ is pseudo-compact iff X is compact.

Coherent groupoids

A groupoid $\mathbb{G}$ is **coherent** if

1. $\mathbb{G}$ is pseudo-compact
2. $\Delta_1: \mathbb{G} \rightarrow \mathbb{G} \times \mathbb{G}$ is pseudo-compact:
for every pseudo-pullback with $\mathbb{C}$ pseudo-compact

$$\begin{array}{ccc} \mathbb{C}' & \longrightarrow & \mathbb{G} \\ \downarrow \lrcorner_{\text{ps}} & & \downarrow \Delta \\ \mathbb{C} & \longrightarrow & \mathbb{G} \times \mathbb{G} \end{array}$$

$\mathbb{C}'$ is also pseudo-compact.

3. $\Delta_2: \Delta_1 \rightarrow \Delta_1 \times_{\mathbb{G} \times \mathbb{G}} \Delta_1$ is pseudo-compact (in the sense above).

NB: A discrete groupoid $\underline{X}$ is coherent iff X is a coherent presheaf.

Recognising coherent groupoids

Lemma

A groupoid $\mathbb{G}$ is coherent if and only if there are a groupoid $\mathbb{K} = (K_1 \rightrightarrows K_0)$ and a (weak) equivalence $\mathbb{K} \rightarrow \mathbb{G}$ such that:

1. K_0 is a compact object: there is P s.t. $yP \twoheadrightarrow K_0$
2. $K_1 \rightarrow K_0 \times K_0$ is compact:

for every K compact and

$$\begin{array}{ccc} K' & \longrightarrow & K_1 \\ \downarrow & \lrcorner & \downarrow \\ K & \longrightarrow & K_0 \times K_0 \end{array} \quad K' \text{ is compact.}$$

3. $K_1 \rightarrow K_1 \times_{K_0 \times K_0} K_1$ is compact.

The Quillen model structure on groupoids

We use the **strong stacks** model structure:

Theorem (Joyal & Tierney)

There is a model structure on $\mathbf{Gpd}(\widehat{\mathcal{P}\mathcal{A}sm}) = [\mathcal{P}\mathcal{A}sm, \mathbf{Gpd}]$ where:

- ▶ The cofibrations are the functors that are objectwise injective on objects.
- ▶ The weak equivalences are the functors that are objectwise equivalences of groupoids (= functors that are fully faithful and essentially surjective on objects.)

Joyal, Tierney. Strong stacks and classifying spaces. 1991

This holds for $\mathbf{Gpd}(\mathcal{E})$ with $\mathcal{E}$ a Grothendieck topos.

For $\mathcal{E} = \mathcal{Set}$, it coincides with the folk model structure on $\mathbf{Gpd}$, but it has less fibrations otherwise (since equivalences $\neq$ strong equivalences in a general $\mathcal{E}$).

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This model structure agrees with the restriction to 1-types of the injective model structure on simplicial presheaves used by Shulman to construct univalent universes in $(\infty, 1)$ -toposes.

Shulman. All $(\infty, 1)$ -toposes have strict univalent universes. 2019

The Quillen model structure on coherent groupoids

Proposition

The two weak factorisation systems from the strong stacks model structure on $\mathbf{Gpd}(\widehat{\mathcal{P}\mathcal{A}sm})$ restrict to $\mathbf{CohGpd}(\widehat{\mathcal{P}\mathcal{A}sm})$.

Follows from:

Lemma

If $\mathbb{F} \rightarrow \mathbb{G}$ is an equivalence of groupoids and $\mathbb{F}$ is coherent, then so is $\mathbb{G}$.

Lemma (Joyal–Tierney)

A trivial fibration is a strong equivalence of groupoids.

The main result

Theorem

Let $\mathbb{G}$ be a coherent groupoid which is also a fibrant o-type.

Then $\pi_o \mathbb{G}$ is coherent (in $\widehat{\mathcal{P}\mathcal{A}sm}$) and $\mathbb{G} \rightarrow \underline{\pi_o \mathbb{G}}$ is a strong equivalence.

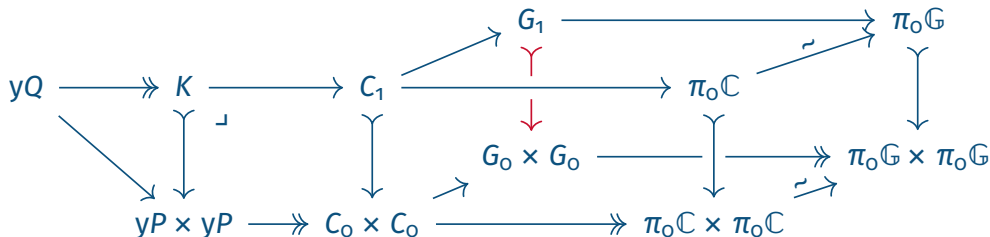
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Recall that $\mathbb{G}$ is coherent iff there is a weak equivalence $\mathbb{C} \rightarrow \mathbb{G}$ with $C_0, C_1 \rightarrow C_0 \times C_0$, and $C_1 \rightarrow C_1 \times_{C_0 \times C_0} C_1$ compact.

If $\mathbb{G}$ is also a fibrant o-type, then $G_1 \rightarrow G_0 \times G_0$ is monic.



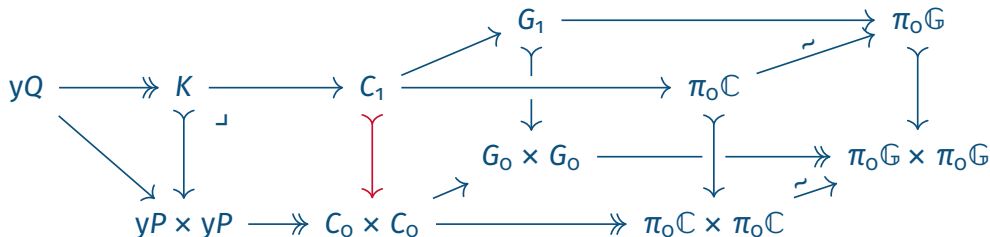
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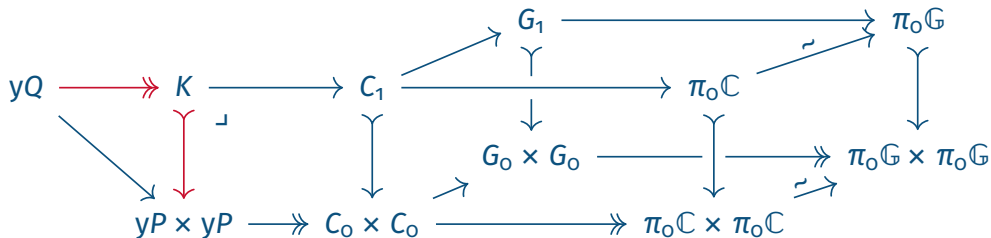
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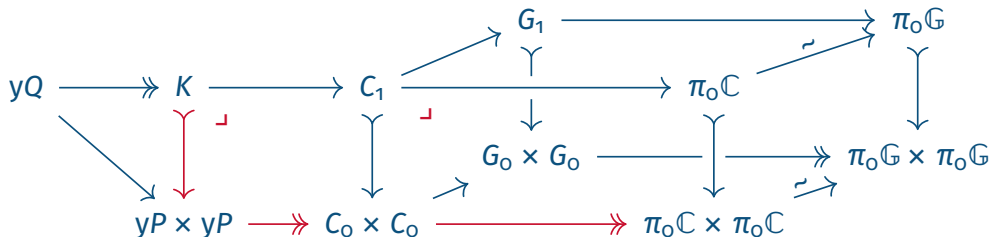
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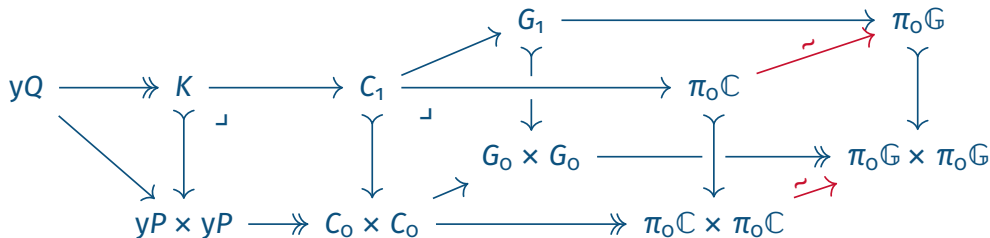
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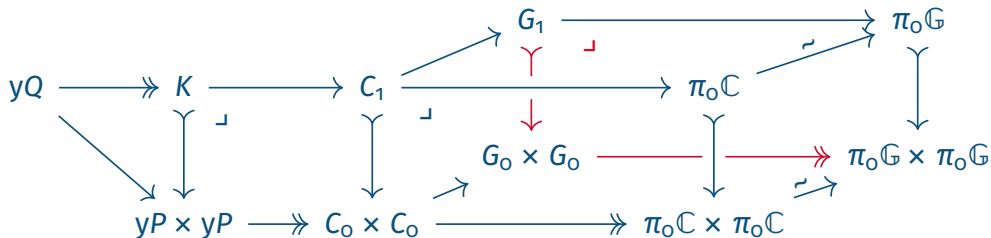
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Corollary

The full subcategory of $\mathbf{CohGpd}(\widehat{\mathcal{P}\mathcal{A}sm})$ on the fibrant o-types is equivalent to the effective topos:

$$\mathbf{CohGpd}(\widehat{\mathcal{P}\mathcal{A}sm})_{o,f} \simeq \mathbf{Coh}(\widehat{\mathcal{P}\mathcal{A}sm}) = \mathcal{E}ff$$

$\mathbf{Eff}_2 := \mathbf{CohGpd}(\widehat{\mathcal{P}\mathcal{A}sm})$ is the restriction to 1-types of the effective ∞ -topos constructed by M. Anel, S. Awodey, R. Barton.

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Thank you!