Intuitionistic K: Proofs and Countermodels

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Synthetic mathematics, logic-affine computation and efficient proof systems

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Two trends in intuitionistic modal logic:

- Intuitionistic modal logic, motivated by meta-logical properties, **IK** [Fischer-Servi, 1984, Simpson, 1994]
- Constructive modal logic, motivated by applications in computer science, **CCDL** [Wijesekera, 1990], **CK** [Bellin et al., 2001]

A semantic approach to IML

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- $x \Vdash A \supset B$ iff $\forall x'.(x \leq x'$ implies $x' \nVdash A$ or $x' \Vdash B)$.
- Monotonic valuation: $x \leq x' \Rightarrow V(x) \subseteq V(x')$;

  Hereditary property (HP): if $x \Vdash A$ and $x \leq y$ then $y \Vdash A$

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Let $\mathcal{M} = (W, R, V)$ and $x, y \in W$

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▷ IML combines both types of language and semantics together

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(FC) forward-confluent $\forall y.x \leq y\ \forall z.Rxz\ \exists u.Ryu\ \&\ z \leq u$

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y & \\
x \arrow[u, "\leq"] \arrow[r, "R"'] \arrow[ur, phantom, "\color{blue}\mathrm{FC}"] & z
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- interpretation of modalities:

  global $\Box$: $\mathcal{M}, x \models \Box A$ iff $\forall y. x \leq y\ \forall z. (Ryz \Rightarrow \mathcal{M}, z \models A)$;

  local $\Diamond$: $\mathcal{M}, x \models \Diamond A$ iff $\exists y. (Rxy\ \&\ \mathcal{M}, y \models A)$.

- Hereditary property of $\Diamond$-formulas is ensured by (FC).

Intuitionistic K: axiomatization [Fischer-Servi, 1984]

Language of **IK**:

$$A ::= p \mid \bot \mid \top \mid (A \vee A) \mid (A \wedge A) \mid (A \supset A) \mid \Box A \mid \Diamond A$$

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Axioms:

k1.	$\Box(A \supset B) \supset (\Box A \supset \Box B)$	$(\mathsf{K}_\Box)$
k2.	$\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$	$(\mathsf{K}_\Diamond)$
k3.	$\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$	(DP)
k4.	$(\Diamond A \supset \Box B) \supset \Box(A \supset B)$	(FS)
k5.	$\Diamond\bot \supset \bot$	(N)

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k5.	$\Diamond\bot \supset \bot$	(N)

Rules:

$$(\mathsf{MP})\ \frac{A \qquad A \supset B}{B} \qquad\qquad (\mathsf{Nec})\ \frac{A}{\Box A}$$

Axiom systems of several IMLs

Axiom system for **IK**:

- $(\mathrm{K}_\Box)$ $\Box(A \supset B) \supset (\Box A \supset \Box B)$
- $(\mathrm{K}_\Diamond)$ $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$
- (MP) (Nec)
- (N) $\neg\Diamond\bot$
- (DP) $\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$
- (FS) $(\Diamond A \supset \Box B) \supset \Box(A \supset B)$

Axiom systems of several IMLs

Axiom system for **LIK** [Balbiani et al., 2024b]:

- $(\mathsf{K}_\Box)$ $\Box(A \supset B) \supset (\Box A \supset \Box B)$
- $(\mathsf{K}_\Diamond)$ $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$
- (MP) (Nec)
- (N) $\neg\Diamond\bot$
- (DP) $\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$
- (CD/RV) $\Box(A \vee B) \supset \Diamond A \vee \Box B$

Axiom systems of several IMLs

Axiom system for **FIK** [Balbiani et al., 2024a]:

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- $(\mathrm{K}_\Diamond)$ $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$
- (MP) (Nec)
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- (DP) $\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$
- (wCD) $\Box(A \vee B) \supset ((\Diamond A \supset \Box B) \supset \Box B)$.

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Axiom system for **CCDL**:

- $(\mathsf{K}_\Box)$ $\Box(A \supset B) \supset (\Box A \supset \Box B)$
- $(\mathsf{K}_\Diamond)$ $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$
- (MP) (Nec)
- (N) $\neg\Diamond\bot$

Axiom systems of several IMLs

Axiom system for **CK**:

- $(\mathrm{K}_\Box)$ $\Box(A \supset B) \supset (\Box A \supset \Box B)$
- $(\mathrm{K}_\Diamond)$ $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$
- (MP) (Nec)

Hierarchy of several IMLs

$$
\begin{tikzcd}
\mathbf{IK} & \mathbf{LIK} \\
\mathbf{FIK} \arrow[u] \arrow[ur] & \\
\mathbf{CCDL}+(\mathrm{DP}) \arrow[u] & \\
\mathbf{CCDL} \arrow[u] & \\
\mathbf{CK} \arrow[u] &
\end{tikzcd}
$$

Proof theory in the literature

Existing sequent calculi for **IK** and its extensions:

Single-conclusion nested calculi	[Straßburger, 2013, Galmiche and Salhi, 2010]
Multi-conclusion nested calculus	[Kuznets and Straßburger, 2019]
Single-conclusion labelled calculi	[Simpson, 1994]
Fully labelled sequent calculi	[Marin et al., 2021, Girlando et al., 2023]
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In [Balbiani et al., 2024a], a **bi-nested** calculus for **FIK** with these features is provided.

▶ Can we provide a similar bi-nested calculus for **IK**?

From sequent to bi-nested sequent

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$tr(S)$:

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[Brünnler, 2009]

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Bi-nested sequent calculus

- two types of nesting are used, roughly

  $[\cdot]$ simulating R, see [Brünnler, 2009, Poggiolesi, 2009]
  $\langle\cdot\rangle$ simulating $\leq$, see [Fitting, 2014, Das and Marin, 2023]

  e.g.

$$\frac{G\{\Gamma \Rightarrow \Delta, \langle A \Rightarrow B\rangle\}}{G\{\Gamma \Rightarrow \Delta, A \supset B\}}\,(\supset_R) \qquad \frac{G\{\Gamma \Rightarrow \Delta, \Diamond A, [\Sigma \Rightarrow \Pi, A]\}}{G\{\Gamma \Rightarrow \Delta, \Diamond A, [\Sigma \Rightarrow \Pi]\}}\,(\Diamond_R)$$

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- $(\Box_R)$ is global while $(\Diamond_R)$ is local

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- interactive rules characterizing (FC) and (BC)

$$\frac{G\{\Gamma \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi, [\Lambda \Rightarrow \Theta^*]\rangle, [\Lambda \Rightarrow \Theta]\}}{G\{\Gamma \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi\rangle, [\Lambda \Rightarrow \Theta]\}}\ (\mathrm{inter}_{\mathrm{fc}})$$

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The bi-nested calculus C_{IK}

$$\frac{}{G\{\Gamma, \bot \Rightarrow \Delta\}}\ (\bot_L) \qquad \frac{}{G\{\Gamma \Rightarrow \top, \Delta\}}\ (\top_R) \qquad \frac{}{G\{\Gamma, p \Rightarrow \Delta, p\}}\ (\mathrm{id})$$

$$\frac{G\{A, B, \Gamma \Rightarrow \Delta\}}{G\{A \wedge B, \Gamma \Rightarrow \Delta\}}\ (\wedge_L) \qquad \frac{G\{\Gamma \Rightarrow \Delta, A\} \qquad G\{\Gamma \Rightarrow \Delta, B\}}{G\{\Gamma \Rightarrow \Delta, A \wedge B\}}\ (\wedge_R)$$

$$\frac{G\{\Gamma, A \Rightarrow \Delta\} \qquad G\{\Gamma, B \Rightarrow \Delta\}}{G\{\Gamma, A \vee B \Rightarrow \Delta\}}\ (\vee_L) \qquad \frac{G\{\Gamma \Rightarrow \Delta, A, B\}}{G\{\Gamma \Rightarrow \Delta, A \vee B\}}\ (\vee_R)$$

$$\frac{G\{\Gamma, A \supset B \Rightarrow A, \Delta\} \qquad G\{\Gamma, B \Rightarrow \Delta\}}{G\{\Gamma, A \supset B \Rightarrow \Delta\}}\ (\supset_L) \qquad \frac{G\{\Gamma \Rightarrow \Delta, \langle A \Rightarrow B\rangle\}}{G\{\Gamma \Rightarrow \Delta, A \supset B\}}\ (\supset_R)$$

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$$\frac{G\{\Gamma \Rightarrow \Delta, [A \Rightarrow]\}}{G\{\Gamma, \Diamond A \Rightarrow \Delta\}}\ (\Diamond_L) \qquad \frac{G\{\Gamma \Rightarrow \Delta, \Diamond A, [\Sigma \Rightarrow \Pi, A]\}}{G\{\Gamma \Rightarrow \Delta, \Diamond A, [\Sigma \Rightarrow \Pi]\}}\ (\Diamond_R)$$

$$\frac{G\{\Gamma, \Gamma' \Rightarrow \Delta, \langle \Gamma', \Sigma \Rightarrow \Pi\rangle\}}{G\{\Gamma, \Gamma' \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi\rangle\}}\ (\mathsf{trans})$$

$$\frac{G\{\Gamma \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi, [\Lambda \Rightarrow \Theta^*]\rangle, [\Lambda \Rightarrow \Theta]\}}{G\{\Gamma \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi\rangle, [\Lambda \Rightarrow \Theta]\}}\ (\mathsf{inter_{fc}})$$

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Example: Proof

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Axiom (FS): $(\Diamond p \supset \Box q) \supset \Box(p \supset q)$ is provable in $\mathbf{C_{IK}}$.

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Axiom (FS): $(\Diamond p \supset \Box q) \supset \Box(p \supset q)$ is provable in $\mathbf{C_{IK}}$.

Let $G_1\{\ \} = \Diamond p \supset \Box q \Rightarrow \langle\{\ \}\rangle$ and $G_2\{\ \} = \Diamond p \supset \Box q \Rightarrow [\Rightarrow \langle p \Rightarrow q\rangle], \langle\{\ \}\rangle$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{G_1\{G_2\{\Diamond p \supset \Box q \Rightarrow \Diamond p, [p \Rightarrow q, p]\}\}}\,(\text{id})}{G_1\{G_2\{\Diamond p \supset \Box q \Rightarrow \Diamond p, [p \Rightarrow q]\}\}}\,(\Diamond_R) \qquad \dfrac{\dfrac{}{G_1\{G_2\{\Box q \Rightarrow [q, p \Rightarrow q]\}\}}\,(\text{id})}{G_1\{G_2\{\Box q \Rightarrow [p \Rightarrow q]\}\}}\,(\Box_L)}{G_1\{G_2\{\Diamond p \supset \Box q \Rightarrow [p \Rightarrow q]\}\}}\,(\supset_L)}{G_1\{\Diamond p \supset \Box q \Rightarrow [\Rightarrow \langle p \Rightarrow q\rangle], \langle \Rightarrow [p \Rightarrow q]\rangle\}}\,(\text{trans})}{G_1\{\Diamond p \supset \Box q \Rightarrow [\Rightarrow \langle p \Rightarrow q\rangle]\}}\,(\text{inter}_{\text{bc}})}{\Diamond p \supset \Box q \Rightarrow \langle \Diamond p \supset \Box q \Rightarrow [\Rightarrow p \supset q]\rangle}\,(\supset_R)}{\Diamond p \supset \Box q \Rightarrow \langle \Rightarrow [\Rightarrow p \supset q]\rangle}\,(\text{trans})}{\Diamond p \supset \Box q \Rightarrow \Box(p \supset q)}\,(\Box_R)
$$

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Let A be a formula. Proof-search for the sequent $\Rightarrow A$ terminates with a finite derivation in which all the leaves are axiomatic or there is one saturated leaf.

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Termination

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Soundess and completeness

$\mathbf{C_{IK}}$ is sound and complete w.r.t. the bi-relational semantics.

Step 1: obtain a saturated leaf

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\hline \\
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\hline \\
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\hline
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\hline
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\UnaryInfC{$(4,6); \stackrel{0}{\Rightarrow} F, \langle \neg\Diamond\neg p \stackrel{1}{\Rightarrow} \Box p, \Diamond\neg p, \langle \neg\Diamond\neg p \stackrel{2}{\Rightarrow} \Diamond\neg p, [\stackrel{3}{\Rightarrow} p, \neg p, \langle p \stackrel{4}{\Rightarrow} \bot \rangle], \langle \stackrel{5}{\Rightarrow} [p \stackrel{6}{\Rightarrow} \bot] \rangle\rangle\rangle$}
\RightLabel{$(\mathrm{inter_{bc}})$}
\UnaryInfC{$\stackrel{0}{\Rightarrow} F, \langle \neg\Diamond\neg p \stackrel{1}{\Rightarrow} \Box p, \Diamond\neg p, \langle \neg\Diamond\neg p \stackrel{2}{\Rightarrow} \Diamond\neg p, [\stackrel{3}{\Rightarrow} p, \neg p, \langle p \stackrel{4}{\Rightarrow} \bot \rangle]\rangle\rangle$}
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\end{prooftree}
$$

We obtain a global-saturated ann-sequent $S_0 = (4,6); \stackrel{0}{\Rightarrow} F, \langle S_1 \rangle$ where S_1 is:

$$\neg\Diamond\neg p \stackrel{1}{\Rightarrow} \Diamond\neg p, \Box p, \langle \neg\Diamond\neg p \stackrel{2}{\Rightarrow} \Diamond\neg p, [\stackrel{3}{\Rightarrow} p, \neg p, \langle p \stackrel{4}{\Rightarrow} \bot \rangle], \langle \neg\Diamond\neg p \stackrel{5}{\Rightarrow} \Diamond\neg p, [p \stackrel{6}{\Rightarrow} \neg p, \bot] \rangle\rangle.$$

Step 2: countermodel construction

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further let

$$S_2 = \neg\Diamond\neg p \overset{2}{\Rightarrow} \Diamond\neg p, [\overset{3}{\Rightarrow} p, \neg p, \langle p \overset{4}{\Rightarrow} \bot \rangle], \langle \neg\Diamond\neg p \overset{5}{\Rightarrow} \Diamond\neg p, [p \overset{6}{\Rightarrow} \neg p, \bot] \rangle$$

$$S_3 = \overset{3}{\Rightarrow} p, \neg p, \langle p \overset{4}{\Rightarrow} \bot \rangle \qquad S_4 = p \overset{4}{\Rightarrow} \bot$$

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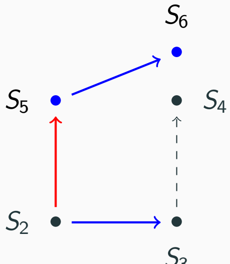

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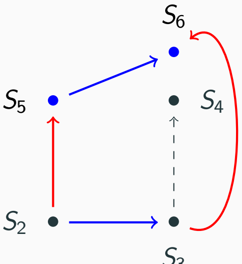

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$$S_2 = \neg\Diamond\neg p \overset{2}{\Rightarrow} \Diamond\neg p, [\overset{3}{\Rightarrow} p, \neg p, \langle p \overset{4}{\Rightarrow} \bot\rangle], \langle \neg\Diamond\neg p \overset{5}{\Rightarrow} \Diamond\neg p, [p \overset{6}{\Rightarrow} \neg p, \bot]\rangle$$

$$S_3 = \overset{3}{\Rightarrow} p, \neg p, \langle p \overset{4}{\Rightarrow} \bot\rangle \qquad S_4 = p \overset{4}{\Rightarrow} \bot$$

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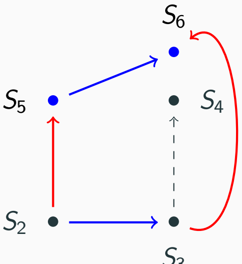

- $W = \{x_{S_0}, x_{S_1}, x_{S_2}, x_{S_3}, x_{S_5}, x_{S_6}\}$
- $x_{S_0} \leq x_{S_1} \leq x_{S_2} \leq x_{S_5}, x_{S_3} \leq x_{S_6}$ (plus reflexivity)
- $Rx_{S_2}x_{S_3}, Rx_{S_5}x_{S_6}$
- $V(x_{S_i}) = \emptyset$ for $i \neq 6$; $V(x_{S_6}) = \{p\}$

Step 2: countermodel construction

$$S_0 = (4,6);\ \overset{0}{\Rightarrow} \neg\Diamond\neg p \supset \Box p, \langle S_1 \rangle$$

where S_1 is:

$\neg\Diamond\neg p \overset{1}{\Rightarrow} \Diamond\neg p, \Box p, \langle \neg\Diamond\neg p \overset{2}{\Rightarrow} \Diamond\neg p, [\overset{3}{\Rightarrow} p, \neg p, \langle p \overset{4}{\Rightarrow} \bot\rangle], \langle \neg\Diamond\neg p \overset{5}{\Rightarrow} \Diamond\neg p, [p \overset{6}{\Rightarrow} \neg p, \bot]\rangle\rangle.$

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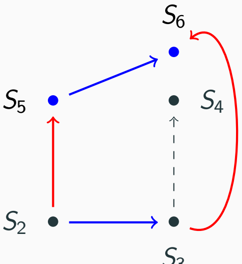

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We can verify $x_{S_0} \not\Vdash \neg\Diamond\neg p \supset \Box p$

Conclusion and perspectives

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- ...

Thanks for your attention!

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Intuitionistic K: Proofs and Countermodels

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