

A Constructive Picture of Noetherianity and Well Quasi-Orders

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Synthetic mathematics, logic-affine computation
and efficient proof systems

Summary

We will see:

- Constructive Noetherian definitions;
- Constructive well quasi-orders and their relations.

Ascending chain condition, classically

Classical logic := Excluded Middle (LEM) + Axiom of Choice (AC).

Classical Noetherianity for Rings

- FBP (Finite Basis Property): every ideal is finitely generated;
- ACC: every ascending chain of ideals *stabilizes*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \Rightarrow \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$

- Classically: $\text{FBP} \leftrightarrow \text{ACC}$.

Problem:

FBP and ACC are not constructively meaningful!

E.g. the 2 element field $\mathbb{F}_2$ is neither constructively FBP nor ACC.

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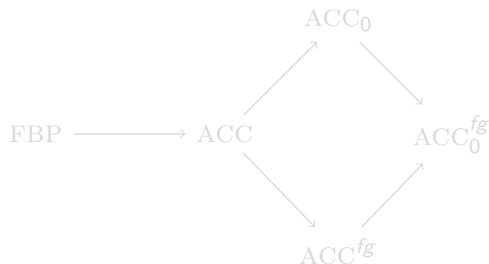
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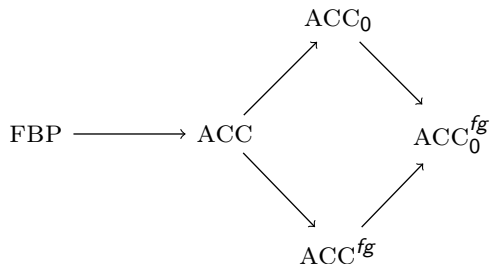
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No other implications*

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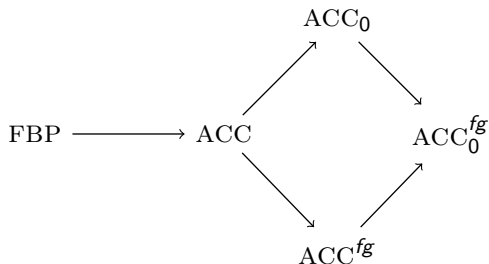
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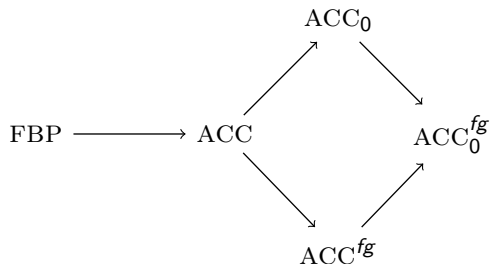
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Auxiliary Properties

Let $(E, \leq)$ be a partial order with $x < y \equiv x \leq y \wedge x \neq y$:

Hereditary conditions

- $H \subseteq E$ is hereditary if $\forall x (\{y \mid y < x\} \subseteq H \Rightarrow x \in H)$;
- E is hereditary well-founded, hwf, if $H \subseteq E$ hereditary $\Rightarrow H = E$;
- E is well ordered if it is hereditary well-founded and linear.

Ascending trees (Richman'03)

An ascending tree in E is a family $(x_i)_{i \in I} \subseteq E$ where

- I is a tree;
- $i < j \Rightarrow x_i \leq x_j$.

An ascending tree *stalls* if $\exists i < j : x_i = x_j$.

Inductive definition of “P bars σ ”

For a predicate P on ascending finite lists on E , we define $P|_{\sigma}$:

- if $P(\sigma)$ then $P|_{\sigma}$;
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- $H \subseteq E$ is hereditary if $\forall x(\{y \mid y < x\} \subseteq H \Rightarrow x \in H)$;
- E is hereditary well-founded, hwf, if $H \subseteq E$ hereditary $\Rightarrow H = E$;
- E is well ordered if it is hereditary well-founded and linear.

Ascending trees (Richman'03)

An **ascending tree** in E is a family $(x_i)_{i \in I} \subseteq E$ where

- I is a tree;
- $i < j \Rightarrow x_i \leq x_j$.

An ascending tree *stalls* if $\exists i < j : x_i = x_j$.

Inductive definition of “P bars σ ”

For a predicate P on ascending finite lists on E , we define $P|\sigma$:

- if $P(\sigma)$ then $P|\sigma$;
- if $P|\sigma x$ for all $x \geq \sigma$, then $P|\sigma$.

Auxiliary Properties

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Intuitionistic Noetherian properties and their relations

A partial order $(E, \leq)$ is

- RS-Noetherian if for $e_1 \leq e_2 \leq \dots$ there is n with $e_n = e_{n+1}$;
- ML-Noetherian if the reverse order $(E, \geq)$ is hwf;
- strongly Noetherian if there is a well-order W and a strictly descending map $\varphi: E \rightarrow W$, i.e. $e < f \Rightarrow \varphi(e) > \varphi(f)$;
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Def: given a ring R , $\mathcal{I}_f(R)$ is the set of finitely generated ideals of R .

Def: a ring R is $*$ Noetherian if $(\mathcal{I}_f(R), \subseteq)$ is $*$ Noetherian.

Constructive implications for a decidable poset $(E, \leq)$



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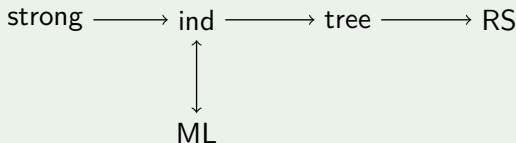
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Basic definitions for quasi-orders

Quasi-order

A qo $(Q, \leq)$ is a set Q with a transitive and reflexive relation $\leq$.

Notation

- $p < q \equiv p \leq q \wedge q \not\leq p$;
- $p \perp q \equiv p \not\leq q \wedge q \not\leq p$;
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Auxiliary definitions

For every qo $(Q, \leq)$:

- the closure of $B \subseteq Q$ is $\uparrow B := \{q \in Q \mid \exists b \in B \ b \leq q\}$;
- B is closed if $B = \uparrow B$ and finitely generated if $B = \uparrow \{b_1, \dots, b_n\}$;
- a sequence $(q_k)_k$ in Q is a total function from $\mathbb{N}$ to Q ;
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A qo $(Q, \leq)$ is

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- wqo if for any sequence $(q_k)_k$ in Q there exist $i < j$ with $q_i \leq q_j$;
- wqo(set) if every sequence $(q_k)_k$ in Q has an infinite ascending subsequence: there are $k_0 < k_1 < \dots$ such that $q_{k_0} \leq q_{k_1} \leq \dots$;
- wqo(anti) if it is well-founded and every antichain is finite;
- wqo(ext) if every linear extension of Q is well-founded;
- wqo(fbp) if every closed subset is finitely generated;
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Remark: all the wqo definitions are classically equivalent.

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- wqo if for any sequence $(q_k)_k$ in Q there exist $i < j$ with $q_i \leq q_j$;
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- **wqo(anti)** if it is well-founded and every antichain is finite;
- wqo(ext) if every linear extension of Q is well-founded;
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Remark: all the wqo definitions are classically equivalent.

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Constructive relations between wqo definitions

Theorem

The conditions $wqo(\text{set})$, $wqo(\text{fbp})$ and $wqo(\text{acc})$ are not constructively meaningful.

Implications between constructive wqo definitions



A closure property

Let $\mathcal{P}$ any of the properties wqo , $wqo(\text{anti})$, $\dots$ except $wqo(\text{ext})$.
If $(Q, \leq)$ has property $\mathcal{P}$ and $P \subseteq Q$, then $(P, \leq)$ has property $\mathcal{P}$.

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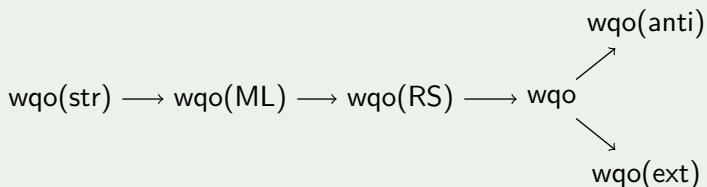
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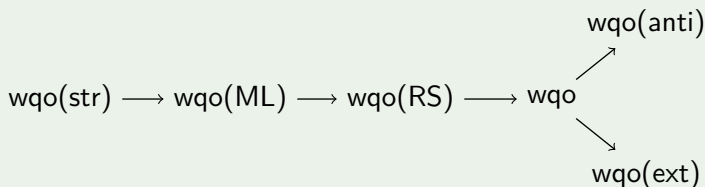
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Future work

Well-founded vs. hereditarily well-founded

Classically equivalent, but not constructively.

Reverse implications

Which of the following implications can be reversed?

- strongly Noetherian $\Rightarrow$ ML-Noetherian;
- $\text{wqo}(\text{RS}) \Rightarrow \text{wqo}$;
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- ...

For now, RS-Noetherian $\not\Rightarrow$ ML-Noetherian by A. Blass.

Further closure properties

Is $\text{wqo}(\text{ext})$ closed under subset?

If P and Q have property $\mathcal{P}$, does

- $P \dot{\cup} Q$ constructively have property $\mathcal{P}$?
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Thank you!

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