A Constructive Picture of Noetherianity and Well Quasi-Orders

Gabriele Buriola[1], Peter Schuster[1], Ingo Blechschmidt[2]

[1]University of Verona, Italy
[2]University of Augsburg, Germany

11/09/2025

Synthetic mathematics, logic-affine computation and efficient proof systems

Summary

We will see:

- Constructive Noetherian definitions;
- Constructive well quasi-orders and their relations.

Ascending chain condition, classically

Classical logic := Excluded Middle (LEM) + Axiom of Choice (AC).

Classical Noetherianity for Rings

- FBP (Finite Basis Property): every ideal is finitely generated;
- ACC: every ascending chain of ideals *stabilizes*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$

- Classically: FBP $\leftrightarrow$ ACC.

Problem:

FBP and ACC are not constructively meaningful!

E.g. the 2 element field $\mathbb{F}_2$ is neither constructively FBP nor ACC.

Ascending chain condition, classically

Classical logic := Excluded Middle (LEM) + Axiom of Choice (AC).

Classical Noetherianity for Rings

- FBP (Finite Basis Property): every ideal is finitely generated;
- ACC: every ascending chain of ideals *stabilizes*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$

- Classically: FBP $\leftrightarrow$ ACC.

Problem:

FBP and ACC are not constructively meaningful!

E.g. the 2 element field $\mathbb{F}_2$ is neither constructively FBP nor ACC.

Ascending chain condition, classically

Classical logic := Excluded Middle (LEM) + Axiom of Choice (AC).

Classical Noetherianity for Rings

- FBP (Finite Basis Property): every ideal is finitely generated;
- ACC: every ascending chain of ideals *stabilizes*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots ;$$

- Classically: FBP $\leftrightarrow$ ACC.

Problem:

FBP and ACC are not constructively meaningful!

E.g. the 2 element field $\mathbb{F}_2$ is neither constructively FBP nor ACC.

Ascending chain condition, classically

Classical logic := Excluded Middle (LEM) + Axiom of Choice (AC).

Classical Noetherianity for Rings

- FBP (Finite Basis Property): every ideal is finitely generated;
- ACC: every ascending chain of ideals *stabilizes*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \ldots;$$

- Classically: FBP $\leftrightarrow$ ACC.

Problem:

FBP and ACC are not constructively meaningful!

E.g. the 2 element field $\mathbb{F}_2$ is neither constructively FBP nor ACC.

Ascending chain condition, classically

Classical logic := Excluded Middle (LEM) + Axiom of Choice (AC).

Classical Noetherianity for Rings

- FBP (Finite Basis Property): every ideal is finitely generated;
- ACC: every ascending chain of ideals *stabilizes*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots ;$$

- Classically: FBP $\leftrightarrow$ ACC.

Problem:

FBP and ACC are not constructively meaningful!

E.g. the 2 element field $\mathbb{F}_2$ is neither constructively FBP nor ACC.

Ascending chain condition, classically

Classical logic := Excluded Middle (LEM) + Axiom of Choice (AC).

Classical Noetherianity for Rings

- FBP (Finite Basis Property): every ideal is finitely generated;
- ACC: every ascending chain of ideals *stabilizes*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$

- Classically: FBP $\leftrightarrow$ ACC.

Problem:

FBP and ACC are not constructively meaningful!

E.g. the 2 element field $\mathbb{F}_2$ is neither constructively FBP nor ACC.

Ascending chain condition, classically

Classical logic := Excluded Middle (LEM) + Axiom of Choice (AC).

Classical Noetherianity for Rings

- FBP (Finite Basis Property): every ideal is finitely generated;
- ACC: every ascending chain of ideals *stabilizes*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots ;$$

- Classically: FBP $\leftrightarrow$ ACC.

Problem:

FBP and ACC are not constructively meaningful!

E.g. the 2 element field $\mathbb{F}_2$ is neither constructively FBP nor ACC.

Ascending chain condition, classically

Classical logic := Excluded Middle (LEM) + Axiom of Choice (AC).

Classical Noetherianity for Rings

- FBP (Finite Basis Property): every ideal is finitely generated;
- ACC: every ascending chain of ideals *stabilizes*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots ;$$

- Classically: FBP $\leftrightarrow$ ACC.

Problem:

FBP and ACC are not constructively meaningful!

E.g. the 2 element field $\mathbb{F}_2$ is neither constructively FBP nor ACC.

Ascending chain conditions, constructively

Toward a constructive ACC

- ACC: every ascending chain of ideals stabilizes
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$
- ACC^{fg}: every ascending chain of finitely generated ideals stabilizes
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$
- ACC_0: every ascending chain of ideals *stalls*
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$
- ACC_0^{fg}: every ascending chain of finitely generated ideals stalls
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$

ACC: is *not* constructive, e.g. $\mathbb{F}_2$;

ACC^{fg}: is *not* constructive, e.g. by Halting problem for Turing machines;

ACC_0: is *not* constructive, e.g. by topological models of intuitionistic logic;

ACC_0^{fg}: is constructive! as discovered by Richman and Seidenberg;

Notation: RS-Noetherian:=ACC_0^{fg} (d'après Richman and Seidenberg).

Ascending chain conditions, constructively

Toward a constructive ACC

- ACC: every ascending chain of ideals stabilizes

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$

- ACC^{fg}: every ascending chain of finitely generated ideals stabilizes

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$

- ACC_0: every ascending chain of ideals *stalls*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$

- ACC_0^{fg}: every ascending chain of finitely generated ideals stalls

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$

ACC: is *not* constructive, e.g. $\mathbb{F}_2$;

ACC^{fg}: is *not* constructive, e.g. by Halting problem for Turing machines;

ACC_0: is *not* constructive, e.g. by topological models of intuitionistic logic;

ACC_0^{fg}: is constructive! as discovered by Richman and Seidenberg;

Notation: RS-Noetherian:=ACC_0^{fg} (d'après Richman and Seidenberg).

Ascending chain conditions, constructively

Toward a constructive ACC

- ACC: every ascending chain of ideals stabilizes

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \ldots;$$

- ACC^{fg}: every ascending chain of finitely generated ideals stabilizes

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \ldots;$$

- ACC_0: every ascending chain of ideals *stalls*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$

- ACC_0^{fg}: every ascending chain of finitely generated ideals stalls

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$

ACC: is *not* constructive, e.g. $\mathbb{F}_2$;

ACC^{fg}: is *not* constructive, e.g. by Halting problem for Turing machines;

ACC_0: is *not* constructive, e.g. by topological models of intuitionistic logic;

ACC_0^{fg}: is constructive! as discovered by Richman and Seidenberg;

Notation: RS-Noetherian:=ACC_0^{fg} (d'après Richman and Seidenberg).

Ascending chain conditions, constructively

Toward a constructive ACC

- ACC: every ascending chain of ideals stabilizes

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \ldots;$$

- ACC^{fg}: every ascending chain of finitely generated ideals stabilizes

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \ldots;$$

- ACC_0: every ascending chain of ideals *stalls*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$

- ACC_0^{fg}: every ascending chain of finitely generated ideals stalls

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$

ACC: is *not* constructive, e.g. $\mathbb{F}_2$;

ACC^{fg}: is *not* constructive, e.g. by Halting problem for Turing machines;

ACC_0: is *not* constructive, e.g. by topological models of intuitionistic logic;

ACC_0^{fg}: is constructive! as discovered by Richman and Seidenberg;

Notation: RS-Noetherian:=ACC_0^{fg} (d'après Richman and Seidenberg).

Ascending chain conditions, constructively

Toward a constructive ACC

- ACC: every ascending chain of ideals stabilizes

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \ldots;$$

- ACC^{fg}: every ascending chain of finitely generated ideals stabilizes

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \ldots;$$

- ACC_0: every ascending chain of ideals *stalls*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$

- ACC_0^{fg}: every ascending chain of finitely generated ideals stalls

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$

ACC: is *not* constructive, e.g. $\mathbb{F}_2$;

ACC^{fg}: is *not* constructive, e.g. by Halting problem for Turing machines;

ACC_0: is *not* constructive, e.g. by topological models of intuitionistic logic;

ACC_0^{fg}: is constructive! as discovered by Richman and Seidenberg;

Notation: RS-Noetherian:=ACC_0^{fg} (d'après Richman and Seidenberg).

Ascending chain conditions, constructively

Toward a constructive ACC

- ACC: every ascending chain of ideals stabilizes

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$

- ACC^{fg}: every ascending chain of finitely generated ideals stabilizes

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$

- ACC_0: every ascending chain of ideals *stalls*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$

- ACC_0^{fg}: every ascending chain of finitely generated ideals stalls

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$

ACC: is *not* constructive, e.g. $\mathbb{F}_2$;

ACC^{fg}: is *not* constructive, e.g. by Halting problem for Turing machines;

ACC_0: is *not* constructive, e.g. by topological models of intuitionistic logic;

ACC_0^{fg}: is constructive! as discovered by Richman and Seidenberg;

Notation: RS-Noetherian:=ACC_0^{fg} (d'après Richman and Seidenberg).

Ascending chain conditions, constructively

Toward a constructive ACC

- ACC: every ascending chain of ideals stabilizes

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$

- ACC^{fg}: every ascending chain of finitely generated ideals stabilizes

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$

- ACC_0: every ascending chain of ideals *stalls*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$

- ACC_0^{fg}: every ascending chain of finitely generated ideals stalls

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$

ACC: is *not* constructive, e.g. $\mathbb{F}_2$;

ACC^{fg}: is *not* constructive, e.g. by Halting problem for Turing machines;

ACC_0: is *not* constructive, e.g. by topological models of intuitionistic logic;

ACC_0^{fg}: is constructive! as discovered by Richman and Seidenberg;

Notation: RS-Noetherian:=ACC_0^{fg} (d'après Richman and Seidenberg).

Ascending chain conditions, constructively

Toward a constructive ACC

- ACC: every ascending chain of ideals stabilizes
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$
- ACC^{fg}: every ascending chain of finitely generated ideals stabilizes
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$
- ACC_0: every ascending chain of ideals *stalls*
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$
- ACC_0^{fg}: every ascending chain of finitely generated ideals stalls
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$

ACC: is *not* constructive, e.g. $\mathbb{F}_2$;

ACC^{fg}: is *not* constructive, e.g. by Halting problem for Turing machines;

ACC_0: is *not* constructive, e.g. by topological models of intuitionistic logic;

ACC_0^{fg}: is constructive! as discovered by Richman and Seidenberg;

Notation: RS-Noetherian:=ACC_0^{fg} (d'après Richman and Seidenberg).

Ascending chain conditions, constructively

Toward a constructive ACC

- ACC: every ascending chain of ideals stabilizes
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \ldots;$$
- ACC^{fg}: every ascending chain of finitely generated ideals stabilizes
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \ldots;$$
- ACC_0: every ascending chain of ideals *stalls*
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$
- ACC_0^{fg}: every ascending chain of finitely generated ideals stalls
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$

ACC: is *not* constructive, e.g. $\mathbb{F}_2$;

ACC^{fg}: is *not* constructive, e.g. by Halting problem for Turing machines;

ACC_0: is *not* constructive, e.g. by topological models of intuitionistic logic;

ACC_0^{fg}: is constructive! as discovered by Richman and Seidenberg;

Notation: RS-Noetherian:=ACC_0^{fg} (d'après Richman and Seidenberg).

Ascending chain conditions, constructively

Toward a constructive ACC

- ACC: every ascending chain of ideals stabilizes
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$
- ACC^{fg}: every ascending chain of finitely generated ideals stabilizes
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$
- ACC_0: every ascending chain of ideals *stalls*
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$
- ACC_0^{fg}: every ascending chain of finitely generated ideals stalls
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$

ACC: is *not* constructive, e.g. $\mathbb{F}_2$;

ACC^{fg}: is *not* constructive, e.g. by Halting problem for Turing machines;

ACC_0: is *not* constructive, e.g. by topological models of intuitionistic logic;

ACC_0^{fg}: is constructive! as discovered by Richman and Seidenberg;

Notation: RS-Noetherian:=ACC_0^{fg} (d'après Richman and Seidenberg).

Ascending chain conditions, constructively

Toward a constructive ACC

- ACC: every ascending chain of ideals stabilizes
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$
- ACC^{fg}: every ascending chain of finitely generated ideals stabilizes
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$
- ACC_0: every ascending chain of ideals *stalls*
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$
- ACC_0^{fg}: every ascending chain of finitely generated ideals stalls
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$

ACC: is *not* constructive, e.g. $\mathbb{F}_2$;

ACC^{fg}: is *not* constructive, e.g. by Halting problem for Turing machines;

ACC_0: is *not* constructive, e.g. by topological models of intuitionistic logic;

ACC_0^{fg}: is constructive! as discovered by Richman and Seidenberg;

Notation: RS-Noetherian:=ACC_0^{fg} (d'après Richman and Seidenberg).

Ascending chain conditions, relations

$$\begin{tikzcd} & & \mathrm{ACC}_0 \arrow[dr] & \\ \mathrm{FBP} \arrow[r] & \mathrm{ACC} \arrow[ur] \arrow[dr] & & \mathrm{ACC}_0^{fg} \\ & & \mathrm{ACC}^{fg} \arrow[ur] & \end{tikzcd}$$

No other implications*

*Except maybe $\mathrm{ACC}_0 \to \mathrm{ACC}^{fg}$

Ascending chain conditions, relations

$$\begin{tikzcd}
 & & \mathrm{ACC}_0 \arrow[dr] & \\
\mathrm{FBP} \arrow[r] & \mathrm{ACC} \arrow[ur] \arrow[dr] & & \mathrm{ACC}_0^{fg} \\
 & & \mathrm{ACC}^{fg} \arrow[ur] &
\end{tikzcd}$$

No other implications*

*Except maybe $\mathrm{ACC}_0 \to \mathrm{ACC}^{fg}$

Ascending chain conditions, relations

$$\begin{tikzcd}
 & & \mathrm{ACC}_0 \arrow[dr] & \\
\mathrm{FBP} \arrow[r] & \mathrm{ACC} \arrow[ur] \arrow[dr] & & \mathrm{ACC}_0^{fg} \\
 & & \mathrm{ACC}^{fg} \arrow[ur] &
\end{tikzcd}$$

No other implications*

*Except maybe $\mathrm{ACC}_0 \to \mathrm{ACC}^{fg}$

Ascending chain conditions, relations

$$\begin{tikzcd}
 & & \mathrm{ACC}_0 \arrow[dr] & \\
\mathrm{FBP} \arrow[r] & \mathrm{ACC} \arrow[ur] \arrow[dr] & & \mathrm{ACC}_0^{fg} \\
 & & \mathrm{ACC}^{fg} \arrow[ur] &
\end{tikzcd}$$

No other implications*

*Except maybe $\mathrm{ACC}_0 \to \mathrm{ACC}^{fg}$

Auxiliary Properties

Let $(E, \leqslant)$ be a partial order with $x < y \equiv x \leqslant y \wedge x \neq y$:

Hereditary conditions

- $H \subseteq E$ is hereditary if $\forall x(\{y \mid y < x\} \subseteq H \Rightarrow x \in H)$;
- E is hereditary well-founded, hwf, if $H \subseteq E$ hereditary $\Rightarrow H = E$;
- E is well ordered if it is hereditary well-founded and linear.

Ascending trees (Richman'03)

An ascending tree in E is a family $(x_i)_{i \in I} \subseteq E$ where

- I is a tree;
- $i < j \Rightarrow x_i \leqslant x_j$.

An ascending tree *stalls* if $\exists\, i < j : x_i = x_j$.

Inductive definition of "P bars σ"

For a predicate P on ascending finite lists on E, we define $P|\sigma$:

- if $P(\sigma)$ then $P|\sigma$;
- if $P|\sigma x$ for all $x \geqslant \sigma$, then $P|\sigma$.

Auxiliary Properties

Let $(E, \leqslant)$ be a partial order with $x < y \equiv x \leqslant y \wedge x \neq y$:

Hereditary conditions

- $H \subseteq E$ is hereditary if $\forall x(\{y \mid y < x\} \subseteq H \Rightarrow x \in H)$;
- E is hereditary well-founded, hwf, if $H \subseteq E$ hereditary $\Rightarrow H = E$;
- E is well ordered if it is hereditary well-founded and linear.

Ascending trees (Richman'03)

An ascending tree in E is a family $(x_i)_{i \in I} \subseteq E$ where

- I is a tree;
- $i < j \Rightarrow x_i \leqslant x_j$.

An ascending tree *stalls* if $\exists\, i < j : x_i = x_j$.

Inductive definition of "P bars σ"

For a predicate P on ascending finite lists on E, we define $P|\sigma$:

- if $P(\sigma)$ then $P|\sigma$;
- if $P|\sigma x$ for all $x \geqslant \sigma$, then $P|\sigma$.

Auxiliary Properties

Let $(E, \leqslant)$ be a partial order with $x < y \equiv x \leqslant y \wedge x \neq y$:

Hereditary conditions

- $H \subseteq E$ is hereditary if $\forall x(\{y \mid y < x\} \subseteq H \Rightarrow x \in H)$;
- E is hereditary well-founded, hwf, if $H \subseteq E$ hereditary $\Rightarrow H = E$;
- E is well ordered if it is hereditary well-founded and linear.

Ascending trees (Richman'03)

An ascending tree in E is a family $(x_i)_{i \in I} \subseteq E$ where

- I is a tree;
- $i < j \Rightarrow x_i \leqslant x_j$.

An ascending tree *stalls* if $\exists\, i < j : x_i = x_j$.

Inductive definition of "P bars σ"

For a predicate P on ascending finite lists on E, we define $P|\sigma$:

- if $P(\sigma)$ then $P|\sigma$;
- if $P|\sigma x$ for all $x \geqslant \sigma$, then $P|\sigma$.

Auxiliary Properties

Let $(E, \leqslant)$ be a partial order with $x < y \equiv x \leqslant y \wedge x \neq y$:

Hereditary conditions

- $H \subseteq E$ is hereditary if $\forall x(\{y \mid y < x\} \subseteq H \Rightarrow x \in H)$;
- E is hereditary well-founded, hwf, if $H \subseteq E$ hereditary $\Rightarrow H = E$;
- E is well ordered if it is hereditary well-founded and linear.

Ascending trees (Richman'03)

An ascending tree in E is a family $(x_i)_{i \in I} \subseteq E$ where

- I is a tree;
- $i < j \Rightarrow x_i \leqslant x_j$.

An ascending tree *stalls* if $\exists\, i < j : x_i = x_j$.

Inductive definition of "P bars σ"

For a predicate P on ascending finite lists on E, we define $P|\sigma$:

- if $P(\sigma)$ then $P|\sigma$;
- if $P|\sigma x$ for all $x \geqslant \sigma$, then $P|\sigma$.

Auxiliary Properties

Let $(E, \leqslant)$ be a partial order with $x < y \equiv x \leqslant y \wedge x \neq y$:

Hereditary conditions

- $H \subseteq E$ is hereditary if $\forall x(\{y \mid y < x\} \subseteq H \Rightarrow x \in H)$;
- E is hereditary well-founded, hwf, if $H \subseteq E$ hereditary $\Rightarrow H = E$;
- E is well ordered if it is hereditary well-founded and linear.

Ascending trees (Richman'03)

An ascending tree in E is a family $(x_i)_{i \in I} \subseteq E$ where

- I is a tree;
- $i < j \Rightarrow x_i \leqslant x_j$.

An ascending tree *stalls* if $\exists\, i < j : x_i = x_j$.

Inductive definition of "P bars σ"

For a predicate P on ascending finite lists on E, we define $P|\sigma$:

- if $P(\sigma)$ then $P|\sigma$;
- if $P|\sigma x$ for all $x \geqslant \sigma$, then $P|\sigma$.

Auxiliary Properties

Let $(E, \leqslant)$ be a partial order with $x < y \equiv x \leqslant y \wedge x \neq y$:

Hereditary conditions

- $H \subseteq E$ is hereditary if $\forall x(\{y \mid y < x\} \subseteq H \Rightarrow x \in H)$;
- E is hereditary well-founded, hwf, if $H \subseteq E$ hereditary $\Rightarrow H = E$;
- E is well ordered if it is hereditary well-founded and linear.

Ascending trees (Richman'03)

An ascending tree in E is a family $(x_i)_{i \in I} \subseteq E$ where

- I is a tree;
- $i < j \Rightarrow x_i \leqslant x_j$.

An ascending tree *stalls* if $\exists\, i < j : x_i = x_j$.

Inductive definition of "P bars σ"

For a predicate P on ascending finite lists on E, we define $P|\sigma$:

- if $P(\sigma)$ then $P|\sigma$;
- if $P|\sigma x$ for all $x \geqslant \sigma$, then $P|\sigma$.

Auxiliary Properties

Let $(E, \leqslant)$ be a partial order with $x < y \equiv x \leqslant y \wedge x \neq y$:

Hereditary conditions

- $H \subseteq E$ is hereditary if $\forall x(\{y \mid y < x\} \subseteq H \Rightarrow x \in H)$;
- E is hereditary well-founded, hwf, if $H \subseteq E$ hereditary $\Rightarrow H = E$;
- E is well ordered if it is hereditary well-founded and linear.

Ascending trees (Richman'03)

An ascending tree in E is a family $(x_i)_{i \in I} \subseteq E$ where

- I is a tree;
- $i < j \Rightarrow x_i \leqslant x_j$.

An ascending tree *stalls* if $\exists\, i < j \,:\, x_i = x_j$.

Inductive definition of "P bars σ"

For a predicate P on ascending finite lists on E, we define $P|\sigma$:

- if $P(\sigma)$ then $P|\sigma$;
- if $P|\sigma x$ for all $x \geqslant \sigma$, then $P|\sigma$.

Auxiliary Properties

Let $(E, \leqslant)$ be a partial order with $x < y \equiv x \leqslant y \wedge x \neq y$:

Hereditary conditions

- $H \subseteq E$ is hereditary if $\forall x(\{y \mid y < x\} \subseteq H \Rightarrow x \in H)$;
- E is hereditary well-founded, hwf, if $H \subseteq E$ hereditary $\Rightarrow H = E$;
- E is well ordered if it is hereditary well-founded and linear.

Ascending trees (Richman'03)

An ascending tree in E is a family $(x_i)_{i \in I} \subseteq E$ where

- I is a tree;
- $i < j \Rightarrow x_i \leqslant x_j$.

An ascending tree *stalls* if $\exists\, i < j \,:\, x_i = x_j$.

Inductive definition of "P bars σ"

For a predicate P on ascending finite lists on E, we define $P|\sigma$:

- if $P(\sigma)$ then $P|\sigma$;
- if $P|\sigma x$ for all $x \geqslant \sigma$, then $P|\sigma$.

Auxiliary Properties

Let $(E, \leqslant)$ be a partial order with $x < y \equiv x \leqslant y \wedge x \neq y$:

Hereditary conditions

- $H \subseteq E$ is hereditary if $\forall x(\{y \mid y < x\} \subseteq H \Rightarrow x \in H)$;
- E is hereditary well-founded, hwf, if $H \subseteq E$ hereditary $\Rightarrow H = E$;
- E is well ordered if it is hereditary well-founded and linear.

Ascending trees (Richman'03)

An ascending tree in E is a family $(x_i)_{i \in I} \subseteq E$ where

- I is a tree;
- $i < j \Rightarrow x_i \leqslant x_j$.

An ascending tree *stalls* if $\exists\, i < j : x_i = x_j$.

Inductive definition of "P bars σ"

For a predicate P on ascending finite lists on E, we define $P|\sigma$:

- if $P(\sigma)$ then $P|\sigma$;
- if $P|\sigma x$ for all $x \geqslant \sigma$, then $P|\sigma$.

Auxiliary Properties

Let $(E, \leqslant)$ be a partial order with $x < y \equiv x \leqslant y \wedge x \neq y$:

Hereditary conditions

- $H \subseteq E$ is hereditary if $\forall x(\{y \mid y < x\} \subseteq H \Rightarrow x \in H)$;
- E is hereditary well-founded, hwf, if $H \subseteq E$ hereditary $\Rightarrow H = E$;
- E is well ordered if it is hereditary well-founded and linear.

Ascending trees (Richman'03)

An ascending tree in E is a family $(x_i)_{i \in I} \subseteq E$ where

- I is a tree;
- $i < j \Rightarrow x_i \leqslant x_j$.

An ascending tree *stalls* if $\exists\, i < j : x_i = x_j$.

Inductive definition of "P bars σ"

For a predicate P on ascending finite lists on E, we define $P|\sigma$:

- if $P(\sigma)$ then $P|\sigma$;
- if $P|\sigma x$ for all $x \geqslant \sigma$, then $P|\sigma$.

Intuitionistic Noetherian properties and their relations

A partial order $(E, \leqslant)$ is

- RS-Noetherian if for $e_1 \leqslant e_2 \leqslant \dots$ there is n with $e_n = e_{n+1}$;
- ML-Noetherian if the reverse order $(E, \geqslant)$ is hwf;
- strongly Noetherian if there is a well-order W and a strictly descending map $\varphi \colon E \to W$, i.e. $e < f \Rightarrow \varphi(e) > \varphi(f)$;
- tree Noetherian if every ascending tree in E stalls;
- inductively Noetherian if Stall $|$ [], where Stall(σ)=“σ is an ascending finite list with repeated terms”.

Def: given a ring R, $\mathcal{I}_f(R)$ is the set of finitely generated ideals of R.
Def: a ring R is * Noetherian if $(\mathcal{I}_f(R), \subseteq)$ is * Noetherian.

Constructive implications for a decidable poset $(E, \leqslant)$

$$\begin{tikzcd}
\text{strong} \arrow[r] & \text{ind} \arrow[r] \arrow[d, leftrightarrow] & \text{tree} \arrow[r] & \text{RS} \\
& \text{ML} & &
\end{tikzcd}$$

Intuitionistic Noetherian properties and their relations

A partial order $(E, \leqslant)$ is

- RS-Noetherian if for $e_1 \leqslant e_2 \leqslant \dots$ there is n with $e_n = e_{n+1}$;
- ML-Noetherian if the reverse order $(E, \geqslant)$ is hwf;
- strongly Noetherian if there is a well-order W and a strictly descending map $\varphi \colon E \to W$, i.e. $e < f \Rightarrow \varphi(e) > \varphi(f)$;
- tree Noetherian if every ascending tree in E stalls;
- inductively Noetherian if $\mathrm{Stall} \mid []$, where $\mathrm{Stall}(\sigma)$=“σ is an ascending finite list with repeated terms”.

Def: given a ring R, $\mathcal{I}_f(R)$ is the set of finitely generated ideals of R.
Def: a ring R is * Noetherian if $(\mathcal{I}_f(R), \subseteq)$ is * Noetherian.

Constructive implications for a decidable poset $(E, \leqslant)$

$$\begin{tikzcd} \text{strong} \arrow[r] & \text{ind} \arrow[r] \arrow[d, leftrightarrow] & \text{tree} \arrow[r] & \text{RS} \\ & \text{ML} & & \end{tikzcd}$$

Intuitionistic Noetherian properties and their relations

A partial order $(E, \leqslant)$ is

- RS-Noetherian if for $e_1 \leqslant e_2 \leqslant \ldots$ there is n with $e_n = e_{n+1}$;
- ML-Noetherian if the reverse order $(E, \geqslant)$ is hwf;
- strongly Noetherian if there is a well-order W and a strictly descending map $\varphi\colon E \to W$, i.e. $e < f \Rightarrow \varphi(e) > \varphi(f)$;
- tree Noetherian if every ascending tree in E stalls;
- inductively Noetherian if Stall | [], where Stall(σ)="σ is an ascending finite list with repeated terms".

Def: given a ring R, $\mathcal{I}_f(R)$ is the set of finitely generated ideals of R.
Def: a ring R is * Noetherian if $(\mathcal{I}_f(R), \subseteq)$ is * Noetherian.

Constructive implications for a decidable poset $(E, \leqslant)$

$$\begin{array}{ccccccc} \text{strong} & \longrightarrow & \text{ind} & \longrightarrow & \text{tree} & \longrightarrow & \text{RS} \\ & & \updownarrow & & & & \\ & & \text{ML} & & & & \end{array}$$

Intuitionistic Noetherian properties and their relations

A partial order $(E, \leqslant)$ is

- RS-Noetherian if for $e_1 \leqslant e_2 \leqslant \ldots$ there is n with $e_n = e_{n+1}$;
- ML-Noetherian if the reverse order $(E, \geqslant)$ is hwf;
- strongly Noetherian if there is a well-order W and a strictly descending map $\varphi\colon E \to W$, i.e. $e < f \Rightarrow \varphi(e) > \varphi(f)$;
- tree Noetherian if every ascending tree in E stalls;
- inductively Noetherian if $\mathrm{Stall} \mid []$, where $\mathrm{Stall}(\sigma)$="σ is an ascending finite list with repeated terms".

Def: given a ring R, $\mathcal{I}_f(R)$ is the set of finitely generated ideals of R.
Def: a ring R is * Noetherian if $(\mathcal{I}_f(R), \subseteq)$ is * Noetherian.

Constructive implications for a decidable poset $(E, \leqslant)$

$$\begin{array}{ccccccc} \text{strong} & \longrightarrow & \text{ind} & \longrightarrow & \text{tree} & \longrightarrow & \text{RS} \\ & & \updownarrow & & & & \\ & & \text{ML} & & & & \end{array}$$

Intuitionistic Noetherian properties and their relations

A partial order $(E, \leqslant)$ is

- RS-Noetherian if for $e_1 \leqslant e_2 \leqslant \dots$ there is n with $e_n = e_{n+1}$;
- ML-Noetherian if the reverse order $(E, \geqslant)$ is hwf;
- strongly Noetherian if there is a well-order W and a strictly descending map $\varphi \colon E \to W$, i.e. $e < f \Rightarrow \varphi(e) > \varphi(f)$;
- tree Noetherian if every ascending tree in E stalls;
- inductively Noetherian if Stall $|$ [], where Stall(σ)="σ is an ascending finite list with repeated terms".

Def: given a ring R, $\mathcal{I}_f(R)$ is the set of finitely generated ideals of R.
Def: a ring R is * Noetherian if $(\mathcal{I}_f(R), \subseteq)$ is * Noetherian.

Constructive implications for a decidable poset $(E, \leqslant)$

$$\begin{array}{ccccccc} \text{strong} & \longrightarrow & \text{ind} & \longrightarrow & \text{tree} & \longrightarrow & \text{RS} \\ & & \updownarrow & & & & \\ & & \text{ML} & & & & \end{array}$$

Intuitionistic Noetherian properties and their relations

A partial order $(E, \leqslant)$ is

- RS-Noetherian if for $e_1 \leqslant e_2 \leqslant \dots$ there is n with $e_n = e_{n+1}$;
- ML-Noetherian if the reverse order $(E, \geqslant)$ is hwf;
- strongly Noetherian if there is a well-order W and a strictly descending map $\varphi\colon E \to W$, i.e. $e < f \Rightarrow \varphi(e) > \varphi(f)$;
- tree Noetherian if every ascending tree in E stalls;
- inductively Noetherian if Stall $|$ [], where Stall(σ)=“σ is an ascending finite list with repeated terms”.

Def: given a ring R, $\mathcal{I}_f(R)$ is the set of finitely generated ideals of R.
Def: a ring R is * Noetherian if $(\mathcal{I}_f(R), \subseteq)$ is * Noetherian.

Constructive implications for a decidable poset $(E, \leqslant)$

$$
\begin{tikzcd}
\text{strong} \arrow[r] & \text{ind} \arrow[r] \arrow[d, leftrightarrow] & \text{tree} \arrow[r] & \text{RS} \\
 & \text{ML} & &
\end{tikzcd}
$$

Intuitionistic Noetherian properties and their relations

A partial order $(E, \leqslant)$ is

- RS-Noetherian if for $e_1 \leqslant e_2 \leqslant \dots$ there is n with $e_n = e_{n+1}$;
- ML-Noetherian if the reverse order $(E, \geqslant)$ is hwf;
- strongly Noetherian if there is a well-order W and a strictly descending map $\varphi\colon E \to W$, i.e. $e < f \Rightarrow \varphi(e) > \varphi(f)$;
- tree Noetherian if every ascending tree in E stalls;
- inductively Noetherian if $\mathrm{Stall} \mid []$, where $\mathrm{Stall}(\sigma)$="σ is an ascending finite list with repeated terms".

Def: given a ring R, $\mathcal{I}_f(R)$ is the set of finitely generated ideals of R.
Def: a ring R is * Noetherian if $(\mathcal{I}_f(R), \subseteq)$ is * Noetherian.

Constructive implications for a decidable poset $(E, \leqslant)$

$$\begin{array}{ccccccc} \text{strong} & \longrightarrow & \text{ind} & \longrightarrow & \text{tree} & \longrightarrow & \text{RS} \\ & & \updownarrow & & & & \\ & & \text{ML} & & & & \end{array}$$

Intuitionistic Noetherian properties and their relations

A partial order $(E, \leqslant)$ is

- RS-Noetherian if for $e_1 \leqslant e_2 \leqslant \dots$ there is n with $e_n = e_{n+1}$;
- ML-Noetherian if the reverse order $(E, \geqslant)$ is hwf;
- strongly Noetherian if there is a well-order W and a strictly descending map $\varphi \colon E \to W$, i.e. $e < f \Rightarrow \varphi(e) > \varphi(f)$;
- tree Noetherian if every ascending tree in E stalls;
- inductively Noetherian if $\mathrm{Stall} \mid []$, where $\mathrm{Stall}(\sigma) =$ "σ is an ascending finite list with repeated terms".

Def: given a ring R, $\mathcal{I}_f(R)$ is the set of finitely generated ideals of R.

Def: a ring R is * Noetherian if $(\mathcal{I}_f(R), \subseteq)$ is * Noetherian.

Constructive implications for a decidable poset $(E, \leqslant)$

$$\begin{tikzcd} \text{strong} \arrow[r] & \text{ind} \arrow[r] \arrow[d, leftrightarrow] & \text{tree} \arrow[r] & \text{RS} \\ & \text{ML} & & \end{tikzcd}$$

Intuitionistic Noetherian properties and their relations

A partial order $(E, \leqslant)$ is

- RS-Noetherian if for $e_1 \leqslant e_2 \leqslant \ldots$ there is n with $e_n = e_{n+1}$;
- ML-Noetherian if the reverse order $(E, \geqslant)$ is hwf;
- strongly Noetherian if there is a well-order W and a strictly descending map $\varphi\colon E \to W$, i.e. $e < f \Rightarrow \varphi(e) > \varphi(f)$;
- tree Noetherian if every ascending tree in E stalls;
- inductively Noetherian if Stall $|$ [], where Stall(σ)="σ is an ascending finite list with repeated terms".

Def: given a ring R, $\mathcal{I}_f(R)$ is the set of finitely generated ideals of R.
Def: a ring R is * Noetherian if $(\mathcal{I}_f(R), \subseteq)$ is * Noetherian.

Constructive implications for a decidable poset $(E, \leqslant)$

$$\begin{array}{ccccccc} \text{strong} & \longrightarrow & \text{ind} & \longrightarrow & \text{tree} & \longrightarrow & \text{RS} \\ & & \updownarrow & & & & \\ & & \text{ML} & & & & \end{array}$$

Intuitionistic Noetherian properties and their relations

A partial order $(E, \leqslant)$ is

- RS-Noetherian if for $e_1 \leqslant e_2 \leqslant \ldots$ there is n with $e_n = e_{n+1}$;
- ML-Noetherian if the reverse order $(E, \geqslant)$ is hwf;
- strongly Noetherian if there is a well-order W and a strictly descending map $\varphi \colon E \to W$, i.e. $e < f \Rightarrow \varphi(e) > \varphi(f)$;
- tree Noetherian if every ascending tree in E stalls;
- inductively Noetherian if Stall $|$ [], where Stall(σ)=“σ is an ascending finite list with repeated terms”.

Def: given a ring R, $\mathcal{I}_f(R)$ is the set of finitely generated ideals of R.
Def: a ring R is * Noetherian if $(\mathcal{I}_f(R), \subseteq)$ is * Noetherian.

Constructive implications for a decidable poset $(E, \leqslant)$

$$\begin{tikzcd} \text{strong} \arrow[r] & \text{ind} \arrow[r] \arrow[d, leftrightarrow] & \text{tree} \arrow[r] & \text{RS} \\ & \text{ML} & & \end{tikzcd}$$

Basic definitions for quasi-orders

Quasi-order

A qo $(Q, \leqslant)$ is a set Q with a transitive and reflexive relation $\leqslant$.

Notation

- $p < q \equiv p \leqslant q \wedge q \nleqslant p$;
- $p \perp q \equiv p \nleqslant q \wedge q \nleqslant p$;
- $p \sim q \equiv p \leqslant q \wedge q \leqslant p$.

Auxiliary definitions

For every qo $(Q, \leqslant)$:

- the closure of $B \subseteq Q$ is $\uparrow B := \{q \in Q \mid \exists b \in B \; b \leqslant q\}$;
- B is closed if $B = \uparrow B$ and finitely generated if $B = \uparrow\{b_1, \ldots, b_n\}$;
- a sequence $(q_k)_k$ in Q is a total function from $\mathbb{N}$ to Q;
- an antichain is a sequence $(q_k)_k$ such that $q_i \perp q_j$ if $i \neq j$;
- an extension of $(Q, \leqslant)$ is a qo $\preccurlyeq$ on Q extending $\leqslant$, i.e., $p \leqslant q \Rightarrow p \preccurlyeq q$ and $p \preccurlyeq q \wedge q \preccurlyeq p \Rightarrow p \sim q$.

Basic definitions for quasi-orders

Quasi-order

A qo $(Q, \leqslant)$ is a set Q with a transitive and reflexive relation $\leqslant$.

Notation

- $p < q \equiv p \leqslant q \wedge q \nleqslant p$;
- $p \perp q \equiv p \nleqslant q \wedge q \nleqslant p$;
- $p \sim q \equiv p \leqslant q \wedge q \leqslant p$.

Auxiliary definitions

For every qo $(Q, \leqslant)$:

- the closure of $B \subseteq Q$ is $\uparrow B := \{q \in Q \mid \exists b \in B \; b \leqslant q\}$;
- B is closed if $B = \uparrow B$ and finitely generated if $B = \uparrow \{b_1, \ldots, b_n\}$;
- a sequence $(q_k)_k$ in Q is a total function from $\mathbb{N}$ to Q;
- an antichain is a sequence $(q_k)_k$ such that $q_i \perp q_j$ if $i \neq j$;
- an extension of $(Q, \leqslant)$ is a qo $\preccurlyeq$ on Q extending $\leqslant$, i.e., $p \leqslant q \Rightarrow p \preccurlyeq q$ and $p \preccurlyeq q \wedge q \preccurlyeq p \Rightarrow p \sim q$.

Basic definitions for quasi-orders

Quasi-order

A qo $(Q, \leqslant)$ is a set Q with a transitive and reflexive relation $\leqslant$.

Notation

- $p < q \;\equiv\; p \leqslant q \;\wedge\; q \nleqslant p$;
- $p \perp q \;\equiv\; p \nleqslant q \;\wedge\; q \nleqslant p$;
- $p \sim q \;\equiv\; p \leqslant q \;\wedge\; q \leqslant p$.

Auxiliary definitions

For every qo $(Q, \leqslant)$:

- the closure of $B \subseteq Q$ is $\uparrow B := \{q \in Q \mid \exists b \in B\ b \leqslant q\}$;
- B is closed if $B = \uparrow B$ and finitely generated if $B = \uparrow\{b_1, \ldots, b_n\}$;
- a sequence $(q_k)_k$ in Q is a total function from $\mathbb{N}$ to Q;
- an antichain is a sequence $(q_k)_k$ such that $q_i \perp q_j$ if $i \neq j$;
- an extension of $(Q, \leqslant)$ is a qo $\preccurlyeq$ on Q extending $\leqslant$, i.e., $p \leqslant q \Rightarrow p \preccurlyeq q$ and $p \preccurlyeq q \wedge q \preccurlyeq p \Rightarrow p \sim q$.

Basic definitions for quasi-orders

Quasi-order

A qo $(Q, \leqslant)$ is a set Q with a transitive and reflexive relation $\leqslant$.

Notation

- $p < q \equiv p \leqslant q \land q \nleqslant p$;
- $p \perp q \equiv p \nleqslant q \land q \nleqslant p$;
- $p \sim q \equiv p \leqslant q \land q \leqslant p$.

Auxiliary definitions

For every qo $(Q, \leqslant)$:

- the closure of $B \subseteq Q$ is $\uparrow B := \{q \in Q \mid \exists b \in B \ b \leqslant q\}$;
- B is closed if $B = \uparrow B$ and finitely generated if $B = \uparrow\{b_1, \ldots, b_n\}$;
- a sequence $(q_k)_k$ in Q is a total function from $\mathbb{N}$ to Q;
- an antichain is a sequence $(q_k)_k$ such that $q_i \perp q_j$ if $i \neq j$;
- an extension of $(Q, \leqslant)$ is a qo $\preccurlyeq$ on Q extending $\leqslant$, i.e., $p \leqslant q \Rightarrow p \preccurlyeq q$ and $p \preccurlyeq q \land q \preccurlyeq p \Rightarrow p \sim q$.

Basic definitions for quasi-orders

Quasi-order

A qo $(Q, \leqslant)$ is a set Q with a transitive and reflexive relation $\leqslant$.

Notation

- $p < q \;\equiv\; p \leqslant q \;\wedge\; q \nleqslant p$;
- $p \perp q \;\equiv\; p \nleqslant q \;\wedge\; q \nleqslant p$;
- $p \sim q \;\equiv\; p \leqslant q \;\wedge\; q \leqslant p$.

Auxiliary definitions

For every qo $(Q, \leqslant)$:

- the closure of $B \subseteq Q$ is $\uparrow B := \{q \in Q \mid \exists b \in B \;\, b \leqslant q\}$;
- B is closed if $B = \uparrow B$ and finitely generated if $B = \uparrow\{b_1, \ldots, b_n\}$;
- a sequence $(q_k)_k$ in Q is a total function from $\mathbb{N}$ to Q;
- an antichain is a sequence $(q_k)_k$ such that $q_i \perp q_j$ if $i \neq j$;
- an extension of $(Q, \leqslant)$ is a qo $\preccurlyeq$ on Q extending $\leqslant$, i.e., $p \leqslant q \Rightarrow p \preccurlyeq q$ and $p \preccurlyeq q \;\wedge\; q \preccurlyeq p \Rightarrow p \sim q$.

Basic definitions for quasi-orders

Quasi-order

A qo $(Q, \leqslant)$ is a set Q with a transitive and reflexive relation $\leqslant$.

Notation

- $p < q \;\equiv\; p \leqslant q \;\wedge\; q \nleqslant p$;
- $p \perp q \;\equiv\; p \nleqslant q \;\wedge\; q \nleqslant p$;
- $p \sim q \;\equiv\; p \leqslant q \;\wedge\; q \leqslant p$.

Auxiliary definitions

For every qo $(Q, \leqslant)$:

- the closure of $B \subseteq Q$ is $\uparrow B := \{q \in Q \mid \exists b \in B \; b \leqslant q\}$;
- B is closed if $B = \uparrow B$ and finitely generated if $B = \uparrow \{b_1, \ldots, b_n\}$;
- a sequence $(q_k)_k$ in Q is a total function from $\mathbb{N}$ to Q;
- an antichain is a sequence $(q_k)_k$ such that $q_i \perp q_j$ if $i \neq j$;
- an extension of $(Q, \leqslant)$ is a qo $\preccurlyeq$ on Q extending $\leqslant$, i.e., $p \leqslant q \Rightarrow p \preccurlyeq q$ and $p \preccurlyeq q \;\wedge\; q \preccurlyeq p \Rightarrow p \sim q$.

Basic definitions for quasi-orders

Quasi-order

A qo $(Q, \leqslant)$ is a set Q with a transitive and reflexive relation $\leqslant$.

Notation

- $p < q \;\equiv\; p \leqslant q \;\wedge\; q \nleqslant p$;
- $p \perp q \;\equiv\; p \nleqslant q \;\wedge\; q \nleqslant p$;
- $p \sim q \;\equiv\; p \leqslant q \;\wedge\; q \leqslant p$.

Auxiliary definitions

For every qo $(Q, \leqslant)$:

- the closure of $B \subseteq Q$ is $\uparrow B := \{q \in Q \mid \exists b \in B \; b \leqslant q\}$;
- B is closed if $B = \uparrow B$ and finitely generated if $B = \uparrow\{b_1, \ldots, b_n\}$;
- a sequence $(q_k)_k$ in Q is a total function from $\mathbb{N}$ to Q;
- an antichain is a sequence $(q_k)_k$ such that $q_i \perp q_j$ if $i \neq j$;
- an extension of $(Q, \leqslant)$ is a qo $\preccurlyeq$ on Q extending $\leqslant$, i.e., $p \leqslant q \Rightarrow p \preccurlyeq q$ and $p \preccurlyeq q \wedge q \preccurlyeq p \Rightarrow p \sim q$.

Basic definitions for quasi-orders

Quasi-order

A qo $(Q, \leqslant)$ is a set Q with a transitive and reflexive relation $\leqslant$.

Notation

- $p < q \;\equiv\; p \leqslant q \;\wedge\; q \nleqslant p$;
- $p \perp q \;\equiv\; p \nleqslant q \;\wedge\; q \nleqslant p$;
- $p \sim q \;\equiv\; p \leqslant q \;\wedge\; q \leqslant p$.

Auxiliary definitions

For every qo $(Q, \leqslant)$:

- the closure of $B \subseteq Q$ is $\uparrow B := \{q \in Q \mid \exists b \in B \; b \leqslant q\}$;
- B is closed if $B = \uparrow B$ and finitely generated if $B = \uparrow\{b_1, \ldots, b_n\}$;
- a sequence $(q_k)_k$ in Q is a total function from $\mathbb{N}$ to Q;
- an antichain is a sequence $(q_k)_k$ such that $q_i \perp q_j$ if $i \neq j$;
- an extension of $(Q, \leqslant)$ is a qo $\preccurlyeq$ on Q extending $\leqslant$, i.e., $p \leqslant q \Rightarrow p \preccurlyeq q$ and $p \preccurlyeq q \wedge q \preccurlyeq p \Rightarrow p \sim q$.

Basic definitions for quasi-orders

Quasi-order

A qo $(Q, \leqslant)$ is a set Q with a transitive and reflexive relation $\leqslant$.

Notation

- $p < q \;\equiv\; p \leqslant q \;\wedge\; q \nleqslant p$;
- $p \perp q \;\equiv\; p \nleqslant q \;\wedge\; q \nleqslant p$;
- $p \sim q \;\equiv\; p \leqslant q \;\wedge\; q \leqslant p$.

Auxiliary definitions

For every qo $(Q, \leqslant)$:

- the closure of $B \subseteq Q$ is $\uparrow B := \{q \in Q \mid \exists b \in B \; b \leqslant q\}$;
- B is closed if $B = \uparrow B$ and finitely generated if $B = \uparrow\{b_1, \ldots, b_n\}$;
- a sequence $(q_k)_k$ in Q is a total function from $\mathbb{N}$ to Q;
- an antichain is a sequence $(q_k)_k$ such that $q_i \perp q_j$ if $i \neq j$;
- an extension of $(Q, \leqslant)$ is a qo $\preccurlyeq$ on Q extending $\leqslant$, i.e., $p \leqslant q \Rightarrow p \preccurlyeq q$ and $p \preccurlyeq q \;\wedge\; q \preccurlyeq p \Rightarrow p \sim q$.

Well quasi-orders definitions

A qo $(Q, \leqslant)$ is

- well-founded if for $q_1 \geqslant q_2 \geqslant \dots$ there is n such that $q_n = q_{n+1}$;
- wqo if for any sequence $(q_k)_k$ in Q there exist $i < j$ with $q_i \leqslant q_j$;
- wqo(set) if every sequence $(q_k)_k$ in Q has an infinite ascending subsequence: there are $k_0 < k_1 < \dots$ such that $q_{k_0} \leqslant q_{k_1} \leqslant \dots$;
- wqo(anti) if it is well-founded and every antichain is finite;
- wqo(ext) if every linear extension of Q is well-founded;
- wqo(fbp) if every closed subset is finitely generated;
- wqo(acc) if the set of closed subsets is Noetherian;
- wqo(*) if the set of finitely generated closed subsets is *Noetherian.

Remark: all the wqo definitions are classically equivalent.

Well quasi-orders definitions

A qo $(Q, \leqslant)$ is

- well-founded if for $q_1 \geqslant q_2 \geqslant \dots$ there is n such that $q_n = q_{n+1}$;
- wqo if for any sequence $(q_k)_k$ in Q there exist $i < j$ with $q_i \leqslant q_j$;
- wqo(set) if every sequence $(q_k)_k$ in Q has an infinite ascending subsequence: there are $k_0 < k_1 < \dots$ such that $q_{k_0} \leqslant q_{k_1} \leqslant \dots$;
- wqo(anti) if it is well-founded and every antichain is finite;
- wqo(ext) if every linear extension of Q is well-founded;
- wqo(fbp) if every closed subset is finitely generated;
- wqo(acc) if the set of closed subsets is Noetherian;
- wqo(*) if the set of finitely generated closed subsets is *Noetherian.

Remark: all the wqo definitions are classically equivalent.

Well quasi-orders definitions

A qo $(Q, \leqslant)$ is

- well-founded if for $q_1 \geqslant q_2 \geqslant \ldots$ there is n such that $q_n = q_{n+1}$;
- wqo if for any sequence $(q_k)_k$ in Q there exist $i < j$ with $q_i \leqslant q_j$;
- wqo(set) if every sequence $(q_k)_k$ in Q has an infinite ascending subsequence: there are $k_0 < k_1 < \ldots$ such that $q_{k_0} \leqslant q_{k_1} \leqslant \ldots$;
- wqo(anti) if it is well-founded and every antichain is finite;
- wqo(ext) if every linear extension of Q is well-founded;
- wqo(fbp) if every closed subset is finitely generated;
- wqo(acc) if the set of closed subsets is Noetherian;
- wqo(*) if the set of finitely generated closed subsets is *Noetherian.

Remark: all the wqo definitions are classically equivalent.

Well quasi-orders definitions

A qo $(Q, \leqslant)$ is

- well-founded if for $q_1 \geqslant q_2 \geqslant \ldots$ there is n such that $q_n = q_{n+1}$;
- wqo if for any sequence $(q_k)_k$ in Q there exist $i < j$ with $q_i \leqslant q_j$;
- wqo(set) if every sequence $(q_k)_k$ in Q has an infinite ascending subsequence: there are $k_0 < k_1 < \ldots$ such that $q_{k_0} \leqslant q_{k_1} \leqslant \ldots$;
- wqo(anti) if it is well-founded and every antichain is finite;
- wqo(ext) if every linear extension of Q is well-founded;
- wqo(fbp) if every closed subset is finitely generated;
- wqo(acc) if the set of closed subsets is Noetherian;
- wqo(*) if the set of finitely generated closed subsets is *Noetherian.

Remark: all the wqo definitions are classically equivalent.

Well quasi-orders definitions

A qo $(Q, \leqslant)$ is

- well-founded if for $q_1 \geqslant q_2 \geqslant \dots$ there is n such that $q_n = q_{n+1}$;
- wqo if for any sequence $(q_k)_k$ in Q there exist $i < j$ with $q_i \leqslant q_j$;
- wqo(set) if every sequence $(q_k)_k$ in Q has an infinite ascending subsequence: there are $k_0 < k_1 < \dots$ such that $q_{k_0} \leqslant q_{k_1} \leqslant \dots$;
- wqo(anti) if it is well-founded and every antichain is finite;
- wqo(ext) if every linear extension of Q is well-founded;
- wqo(fbp) if every closed subset is finitely generated;
- wqo(acc) if the set of closed subsets is Noetherian;
- wqo(*) if the set of finitely generated closed subsets is *Noetherian.

Remark: all the wqo definitions are classically equivalent.

Well quasi-orders definitions

A qo $(Q, \leqslant)$ is

- well-founded if for $q_1 \geqslant q_2 \geqslant \ldots$ there is n such that $q_n = q_{n+1}$;
- wqo if for any sequence $(q_k)_k$ in Q there exist $i < j$ with $q_i \leqslant q_j$;
- wqo(set) if every sequence $(q_k)_k$ in Q has an infinite ascending subsequence: there are $k_0 < k_1 < \ldots$ such that $q_{k_0} \leqslant q_{k_1} \leqslant \ldots$;
- wqo(anti) if it is well-founded and every antichain is finite;
- wqo(ext) if every linear extension of Q is well-founded;
- wqo(fbp) if every closed subset is finitely generated;
- wqo(acc) if the set of closed subsets is Noetherian;
- wqo(*) if the set of finitely generated closed subsets is *Noetherian.

Remark: all the wqo definitions are classically equivalent.

Well quasi-orders definitions

A qo $(Q, \leqslant)$ is

- well-founded if for $q_1 \geqslant q_2 \geqslant \ldots$ there is n such that $q_n = q_{n+1}$;
- wqo if for any sequence $(q_k)_k$ in Q there exist $i < j$ with $q_i \leqslant q_j$;
- wqo(set) if every sequence $(q_k)_k$ in Q has an infinite ascending subsequence: there are $k_0 < k_1 < \ldots$ such that $q_{k_0} \leqslant q_{k_1} \leqslant \ldots$;
- wqo(anti) if it is well-founded and every antichain is finite;
- wqo(ext) if every linear extension of Q is well-founded;
- wqo(fbp) if every closed subset is finitely generated;
- wqo(acc) if the set of closed subsets is Noetherian;
- wqo(*) if the set of finitely generated closed subsets is *Noetherian.

Remark: all the wqo definitions are classically equivalent.

Well quasi-orders definitions

A qo $(Q, \leqslant)$ is

- well-founded if for $q_1 \geqslant q_2 \geqslant \ldots$ there is n such that $q_n = q_{n+1}$;
- wqo if for any sequence $(q_k)_k$ in Q there exist $i < j$ with $q_i \leqslant q_j$;
- wqo(set) if every sequence $(q_k)_k$ in Q has an infinite ascending subsequence: there are $k_0 < k_1 < \ldots$ such that $q_{k_0} \leqslant q_{k_1} \leqslant \ldots$;
- wqo(anti) if it is well-founded and every antichain is finite;
- wqo(ext) if every linear extension of Q is well-founded;
- wqo(fbp) if every closed subset is finitely generated;
- wqo(acc) if the set of closed subsets is Noetherian;
- wqo(*) if the set of finitely generated closed subsets is *Noetherian.

Remark: all the wqo definitions are classically equivalent.

Well quasi-orders definitions

A qo $(Q, \leqslant)$ is

- well-founded if for $q_1 \geqslant q_2 \geqslant \dots$ there is n such that $q_n = q_{n+1}$;
- wqo if for any sequence $(q_k)_k$ in Q there exist $i < j$ with $q_i \leqslant q_j$;
- wqo(set) if every sequence $(q_k)_k$ in Q has an infinite ascending subsequence: there are $k_0 < k_1 < \dots$ such that $q_{k_0} \leqslant q_{k_1} \leqslant \dots$;
- wqo(anti) if it is well-founded and every antichain is finite;
- wqo(ext) if every linear extension of Q is well-founded;
- wqo(fbp) if every closed subset is finitely generated;
- wqo(acc) if the set of closed subsets is Noetherian;
- wqo(*) if the set of finitely generated closed subsets is *Noetherian.

Remark: all the wqo definitions are classically equivalent.

Well quasi-orders definitions

A qo $(Q, \leqslant)$ is

- well-founded if for $q_1 \geqslant q_2 \geqslant \dots$ there is n such that $q_n = q_{n+1}$;
- wqo if for any sequence $(q_k)_k$ in Q there exist $i < j$ with $q_i \leqslant q_j$;
- wqo(set) if every sequence $(q_k)_k$ in Q has an infinite ascending subsequence: there are $k_0 < k_1 < \dots$ such that $q_{k_0} \leqslant q_{k_1} \leqslant \dots$;
- wqo(anti) if it is well-founded and every antichain is finite;
- wqo(ext) if every linear extension of Q is well-founded;
- wqo(fbp) if every closed subset is finitely generated;
- wqo(acc) if the set of closed subsets is Noetherian;
- wqo(*) if the set of finitely generated closed subsets is *Noetherian.

Remark: all the wqo definitions are classically equivalent.

Well quasi-orders definitions

A qo $(Q, \leqslant)$ is

- well-founded if for $q_1 \geqslant q_2 \geqslant \dots$ there is n such that $q_n = q_{n+1}$;
- wqo if for any sequence $(q_k)_k$ in Q there exist $i < j$ with $q_i \leqslant q_j$;
- wqo(set) if every sequence $(q_k)_k$ in Q has an infinite ascending subsequence: there are $k_0 < k_1 < \dots$ such that $q_{k_0} \leqslant q_{k_1} \leqslant \dots$;
- wqo(anti) if it is well-founded and every antichain is finite;
- wqo(ext) if every linear extension of Q is well-founded;
- wqo(fbp) if every closed subset is finitely generated;
- wqo(acc) if the set of closed subsets is Noetherian;
- wqo(*) if the set of finitely generated closed subsets is *Noetherian.

Remark: all the wqo definitions are classically equivalent.

Constructive relations between wqo definitions

Theorem

The conditions wqo(set), wqo(fbp) and wqo(acc) are not constructively meaningful.

Implications between constructive wqo definitions

$$\begin{tikzcd} & & & & \text{wqo(anti)} \\ \text{wqo(str)} \arrow[r] & \text{wqo(ML)} \arrow[r] & \text{wqo(RS)} \arrow[r] & \text{wqo} \arrow[ur] \arrow[dr] & \\ & & & & \text{wqo(ext)} \end{tikzcd}$$

A closure property

Let $\mathcal{P}$ any of the properties wqo, wqo(anti), ... except wqo(ext). If $(Q, \leqslant)$ has property $\mathcal{P}$ and $P \subseteq Q$, then $(P, \leqslant)$ has property $\mathcal{P}$.

Constructive relations between wqo definitions

Theorem

The conditions wqo(set), wqo(fbp) and wqo(acc) are not constructively meaningful.

Implications between constructive wqo definitions

$$\begin{tikzcd}
 & & & & \text{wqo(anti)} \\
\text{wqo(str)} \arrow[r] & \text{wqo(ML)} \arrow[r] & \text{wqo(RS)} \arrow[r] & \text{wqo} \arrow[ur] \arrow[dr] & \\
 & & & & \text{wqo(ext)}
\end{tikzcd}$$

A closure property

Let $\mathcal{P}$ any of the properties wqo, wqo(anti), ...except wqo(ext). If $(Q, \leqslant)$ has property $\mathcal{P}$ and $P \subseteq Q$, then $(P, \leqslant)$ has property $\mathcal{P}$.

Constructive relations between wqo definitions

Theorem

The conditions wqo(set), wqo(fbp) and wqo(acc) are not constructively meaningful.

Implications between constructive wqo definitions

$$\begin{tikzcd}
 & & & & \mathrm{wqo(anti)} \\
\mathrm{wqo(str)} \arrow[r] & \mathrm{wqo(ML)} \arrow[r] & \mathrm{wqo(RS)} \arrow[r] & \mathrm{wqo} \arrow[ur] \arrow[dr] & \\
 & & & & \mathrm{wqo(ext)}
\end{tikzcd}$$

A closure property

Let $\mathcal{P}$ any of the properties wqo, wqo(anti), ... except wqo(ext). If $(Q, \leqslant)$ has property $\mathcal{P}$ and $P \subseteq Q$, then $(P, \leqslant)$ has property $\mathcal{P}$.

Constructive relations between wqo definitions

Theorem

The conditions wqo(set), wqo(fbp) and wqo(acc) are not constructively meaningful.

Implications between constructive wqo definitions

$$\begin{tikzcd}[column sep=small] & & & & \text{wqo(anti)} \\ \text{wqo(str)} \arrow[r] & \text{wqo(ML)} \arrow[r] & \text{wqo(RS)} \arrow[r] & \text{wqo} \arrow[ur] \arrow[dr] & \\ & & & & \text{wqo(ext)} \end{tikzcd}$$

A closure property

Let $\mathcal{P}$ any of the properties wqo, wqo(anti), . . . except wqo(ext).
If $(Q, \leqslant)$ has property $\mathcal{P}$ and $P \subseteq Q$, then $(P, \leqslant)$ has property $\mathcal{P}$.

Future work

Well-founded vs. hereditarily well-founded

Classically equivalent, but not constructively.

Reverse implications

Which of the following implications can be reversed?

- strongly Noetherian $\Rightarrow$ ML-Noetherian;
- wqo(RS) $\Rightarrow$ wqo;
- wqo $\Rightarrow$ wqo(anti);
- ...

For now, RS-Noetherian $\not\Rightarrow$ ML-Noetherian by A. Blass.

Further closure properties

Is wqo(ext) closed under subset?

If P and Q have property $\mathcal{P}$, does

- $P \dot{\cup} Q$ constructively have property $\mathcal{P}$?
- $P \times Q$ constructively have property $\mathcal{P}$?

Future work

Well-founded vs. hereditarily well-founded

Classically equivalent, but not constructively.

Reverse implications

Which of the following implications can be reversed?

- strongly Noetherian $\Rightarrow$ ML-Noetherian;
- wqo(RS) $\Rightarrow$ wqo;
- wqo $\Rightarrow$ wqo(anti);
- ...

For now, RS-Noetherian $\not\Rightarrow$ ML-Noetherian by A. Blass.

Further closure properties

Is wqo(ext) closed under subset?

If P and Q have property $\mathcal{P}$, does

- $P \dot\cup Q$ constructively have property $\mathcal{P}$?
- $P \times Q$ constructively have property $\mathcal{P}$?

Future work

Well-founded vs. hereditarily well-founded

Classically equivalent, but not constructively.

Reverse implications

Which of the following implications can be reversed?

- strongly Noetherian $\Rightarrow$ ML-Noetherian;
- wqo(RS) $\Rightarrow$ wqo;
- wqo $\Rightarrow$ wqo(anti);
- ...

For now, RS-Noetherian $\not\Rightarrow$ ML-Noetherian by A. Blass.

Further closure properties

Is wqo(ext) closed under subset?

If P and Q have property $\mathcal{P}$, does

- $P \dot\cup Q$ constructively have property $\mathcal{P}$?
- $P \times Q$ constructively have property $\mathcal{P}$?

Future work

Well-founded vs. hereditarily well-founded

Classically equivalent, but not constructively.

Reverse implications

Which of the following implications can be reversed?

- strongly Noetherian $\Rightarrow$ ML-Noetherian;
- wqo(RS) $\Rightarrow$ wqo;
- wqo $\Rightarrow$ wqo(anti);
- ...

For now, RS-Noetherian $\not\Rightarrow$ ML-Noetherian by A. Blass.

Further closure properties

Is wqo(ext) closed under subset?

If P and Q have property $\mathcal{P}$, does

- $P \dot\cup Q$ constructively have property $\mathcal{P}$?
- $P \times Q$ constructively have property $\mathcal{P}$?

Future work

Well-founded vs. hereditarily well-founded

Classically equivalent, but not constructively.

Reverse implications

Which of the following implications can be reversed?

- strongly Noetherian $\Rightarrow$ ML-Noetherian;
- wqo(RS) $\Rightarrow$ wqo;
- wqo $\Rightarrow$ wqo(anti);
- ...

For now, RS-Noetherian $\not\Rightarrow$ ML-Noetherian by A. Blass.

Further closure properties

Is wqo(ext) closed under subset?

If P and Q have property $\mathcal{P}$, does

- $P \dot\cup Q$ constructively have property $\mathcal{P}$?
- $P \times Q$ constructively have property $\mathcal{P}$?

Future work

Well-founded vs. hereditarily well-founded

Classically equivalent, but not constructively.

Reverse implications

Which of the following implications can be reversed?

- strongly Noetherian $\Rightarrow$ ML-Noetherian;
- wqo(RS) $\Rightarrow$ wqo;
- wqo $\Rightarrow$ wqo(anti);
- ...

For now, RS-Noetherian $\not\Rightarrow$ ML-Noetherian by A. Blass.

Further closure properties

Is wqo(ext) closed under subset?

If P and Q have property $\mathcal{P}$, does

- $P \mathbin{\dot\cup} Q$ constructively have property $\mathcal{P}$?
- $P \times Q$ constructively have property $\mathcal{P}$?

Future work

Well-founded vs. hereditarily well-founded

Classically equivalent, but <u>not</u> constructively.

Reverse implications

Which of the following implications can be reversed?

- strongly Noetherian $\Rightarrow$ ML-Noetherian;
- wqo(RS) $\Rightarrow$ wqo;
- wqo $\Rightarrow$ wqo(anti);
- ...

For now, RS-Noetherian $\not\Rightarrow$ ML-Noetherian by A. Blass.

Further closure properties

Is wqo(ext) closed under subset?

If P and Q have property $\mathcal{P}$, does

- $P \dot\cup Q$ constructively have property $\mathcal{P}$?
- $P \times Q$ constructively have property $\mathcal{P}$?

Future work

Well-founded vs. hereditarily well-founded

Classically equivalent, but not constructively.

Reverse implications

Which of the following implications can be reversed?

- strongly Noetherian $\Rightarrow$ ML-Noetherian;
- wqo(RS) $\Rightarrow$ wqo;
- wqo $\Rightarrow$ wqo(anti);
- ...

For now, RS-Noetherian $\not\Leftarrow$ ML-Noetherian by A. Blass.

Further closure properties

Is wqo(ext) closed under subset?

If P and Q have property $\mathcal{P}$, does

- $P \dot\cup Q$ constructively have property $\mathcal{P}$?
- $P \times Q$ constructively have property $\mathcal{P}$?

Future work

Well-founded vs. hereditarily well-founded

Classically equivalent, but not constructively.

Reverse implications

Which of the following implications can be reversed?

- strongly Noetherian $\Rightarrow$ ML-Noetherian;
- wqo(RS) $\Rightarrow$ wqo;
- wqo $\Rightarrow$ wqo(anti);
- ...

For now, RS-Noetherian $\not\Rightarrow$ ML-Noetherian by A. Blass.

Further closure properties

Is wqo(ext) closed under subset?

If P and Q have property $\mathcal{P}$, does

- $P \dot{\cup} Q$ constructively have property $\mathcal{P}$?
- $P \times Q$ constructively have property $\mathcal{P}$?

Thank you! References:

- Higman, G.: *Ordering by divisibility in abstract algebras*. Proc. London Math. Soc. 3, 2:326-336, (1952).
- Richman, F.: *Constructive Aspects of Noetherian Rings*. Proceedings of the American Mathematical Society 44(2), 436-441, (1974).
- Seidenberg, A.: *What is Noetherian?* Rendiconti del Seminario matematico e fisico di Milano 44(1), 55-61 (1974).
- Jacobsson C., Löfwall C.: *Standard bases for general coefficient rings and a new constructive proof of Hilbert's basis theorem*. J. Symbolic Comput. 12(3), 337-371 (1991)
- Richman, F.: *The ascending tree condition: constructive algebra without countable choice*. Commun. Algebra 31(4), 1993-2002 (2003)
- Perdry, H.: *Strongly Noetherian rings and constructive ideal theory*. Journal of symbolic computation 37(4), 511-535 (2004).
- Veldman, W.: *An Intuitionistic Proof of Kruskal's Theorem*. Archive for Mathematical Logic 43(2), 215-264 (2004).
- Cholak P., Marcone A., and Solomon R.: *Reverse mathematics and the equivalence of definitions for well and better quasi-orders*. The Journal of symbolic logic, 66(1):683-55, (2004).