Internal languages of locally cartesian closed $(\infty, 1)$-categories

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Outline

From relative categories to quasicategories

Relative categories

$$\mathbf{RelCat}$$

and quasicategories

$$\mathbf{QCat}$$

are both objects that combine homotopy and category. They are in fact equivalent framework to talk about $(\infty, 1)$-categories.

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Definition

The $\mathbf{Ho}_\infty$ functor is defined as the following composite:

$$\begin{tikzcd}
\mathbf{RelCat} \arrow[d, "\text{simplicial localization}"] \\
\mathbf{Cat}_\Delta \arrow[d, "\text{fibrant replacement}"] \\
\mathbf{Cat}_\Delta \arrow[d, "\mathbf{N}_\Delta\text{(simplicial/homotopy-coherent nerve)}"] \\
\mathbf{QCat}
\end{tikzcd}$$

From relative categories to quasicategories

Theorem (Barwick-Kan[1])

The functor

$$\mathbf{Ho}_\infty : \mathbf{RelCat} \to \mathbf{QCat}$$

is a DK-equivalence.

[1] Clark Barwick and Daniel M Kan. "Relative categories: another model for the homotopy theory of homotopy theories". In: *Indagationes Mathematicae* 23.1-2 (2012), pp. 42–68, Clark Barwick and Daniël M Kan. "A Thomason-like Quillen equivalence between quasi-categories and relative categories". In: *arXiv preprint arXiv:1101.0772* (2011).

Other $(\infty, 1)$-categories frameworks

- Model categories
- Fibration/cofibration categories
- **Kan**-enriched categories
- Complete Segal Spaces

The conjectures

Conjecture (Internal language conjectures[2])

The functor $\mathbf{Ho}_\infty$ *restricts to DK-equivalences*

$$\mathbf{CompCat}_{\Sigma,Id} \to \mathbf{Qcat}_{lex}{}^{[3]}$$
$$\mathbf{CompCat}_{\Sigma,\Pi_{ext},Id} \to \mathbf{Qcat}_{lcc}$$

[2] Krzysztof Kapulkin and Peter LeFanu Lumsdaine. "The homotopy theory of type theories". In: *Advances in Mathematics* 337 (2018), pp. 1–38.

[3] Krzysztof Kapulkin and Karol Szumiło. "Internal languages of finitely complete $(\infty,1)$-categories". In: *Selecta Mathematica* 25.2 (2019), pp. 1–46

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where the domain (resp. codomain) categories have morphisms that preserves the structure involved up to isomorphism (resp. equivalence).

[2] Krzysztof Kapulkin and Peter LeFanu Lumsdaine. "The homotopy theory of type theories". In: *Advances in Mathematics* 337 (2018), pp. 1–38.

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Outline

Clans

The notion of clan[4] essentially axiomatizes/rephrases the structure on a category $\mathcal{C}$ equipped with a full comprehension structure, which is needed to interpret the core of type theory.

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Clans

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Definition

A clan structure on a category $\mathcal{C}$ with a terminal object $\mathbf{1}$ is given by a class of maps $\mathcal{F}$ called fibrations such that:

- Isomorphims are fibrations, $X \to \mathbf{1}$ is a fibration for every X.
- Fibration are closed under composition. Pullbacks of fibrations exists and yield fibrations.

[4] André Joyal. "Notes on clans and tribes". In: *arXiv preprint arXiv:1710.10238* (2017).

Tribes

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We will also consider π-tribes, which are essentially tribes such that every fibration admits an *internal product* along any fibration.
Essentially, this means that the underlying type theory also has Π-types.

Canonical comprehension

Given a π-tribe $\mathcal{T}$, the canonical comprehension structure given by the Grothendieck fibration

$$\mathbf{cod} : \mathcal{T}_{\mathsf{fib}}^{\rightarrow} \to \mathcal{T}$$

supports Σ- and Π-types that are stable under pullback up to isomorphism.

Substitution is well-defined and functorial up to isomorphism.

Strictification

A **strictification**[5] procedure aims at replacing this comprehension category by an equivalent split one, and such that Σ- and Π-types are strictly stable under substitution.

Pictorially:

$$\text{Isomorphisms} \quad \Longrightarrow \quad \text{Equalities}$$

[5] Peter LeFanu Lumsdaine and Michael A Warren. "The local universes model: an overlooked coherence construction for dependent type theories". In: *ACM Transactions on Computational Logic (TOCL)* 16.3 (2015), pp. 1–31.

Coherence in an ∞-category

Consider a locally cartesian closed $(\infty,1)$-category $\mathcal{C}$ (e.g. a quasicategory) with finite limits.

- Pullbacks give a substitution operation which is well-defined and functorial up to (homotopy) equivalence.
- Dependent products are also defined (and pullback-stable) up to equivalence.

Rigidification

A **rigidification** procedure aims at replacing $\mathcal{C}$ by a π-tribe presenting the same $(\infty,1)$-category (up to equivalence).

Pictorially:

$$\text{Equivalences} \quad \Longrightarrow \quad \text{Isomorphisms}$$

Outline

DK-equivalences from equivalence of categories

Theorem (Cisinski[6])

Given fibration categories $\mathcal{F}_0$ and $\mathcal{F}_1$, as well as an exact functor $H : \mathcal{F}_0 \to \mathcal{F}_1$, the following are equivalent:

- *H is a DK-equivalence.*
- $\mathbf{Ho}(H) : \mathbf{Ho}(\mathcal{F}_0) \to \mathbf{Ho}(\mathcal{F}_1)$ *is an equivalence of categories.*

[6] Denis-Charles Cisinski. "Catégories dérivables". In: *Bulletin de la société mathématique de France* 138.3 (2010), pp. 317–393.

Fibration category of tribes

Trb can be equipped with a notion of fibration making it "almost" a fibration category in that, given a tribe $\mathcal{T}$, there is a canonical tribe $P\mathcal{T}$ whose objects Reed fibrant homotopical spans in $\mathcal{T}$.

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Trb can be equipped with a notion of fibration making it "almost" a fibration category in that, given a tribe $\mathcal{T}$, there is a canonical tribe $P\mathcal{T}$ whose objects Reed fibrant homotopical spans in $\mathcal{T}$. For every object x, we may chose a path object px (in $\mathcal{T}$) for x:

$$\begin{tikzcd}
 & & x \\
x \arrow[r, dashed, "\sim" description] & px \arrow[ur, two heads, "\sim" description] \arrow[dr, two heads, "\sim" description] & \\
 & & x
\end{tikzcd}$$

Fibration category of tribes

We then have a mapping $i : \mathcal{T} \to P\mathcal{T}$ fitting in a commutative triangle

$$\begin{tikzcd} & P\mathcal{T} \arrow[dr, "<\pi_1{,}\pi_2>"] & \\ \mathcal{T} \arrow[ur, "i"] \arrow[rr, "\Delta"'] & & \mathcal{T} \times \mathcal{T} \end{tikzcd}$$

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However, the choice made need not imply that the mapping i is functorial. This is how we fall short of constructing a path object for $\mathcal{T}$, the only thing left needed for **Trb** to be a fibration category.
If $\mathcal{T}$ is a semi-cubical tribe, taking $Px := x^{\square^1}$ makes it functorial, and we write $\iota_{\mathcal{T}} : \mathcal{T} \to P\mathcal{T}$

The approach

To prove the second part of the conjecture, our approach is the following:

$$
\begin{tikzcd}
 & \mathbf{scTrb}_{\pi} \arrow[d, "\sim"] & \mathbf{scTrb}^{p}_{\pi} \arrow[l, "\sim"'] \arrow[dd, "\sim"] \\
\mathbf{CompCat}_{\Sigma,\Pi_{\mathrm{ext}},\mathrm{Id}} \arrow[r, "\sim"] \arrow[d] & \mathbf{Trb}_{\pi} \arrow[d] & \\
\mathbf{QCat}_{lcc} & \mathbf{Trb}_{\pi,\sim} \arrow[l, "\sim"'] & \mathbf{scTrb}_{\pi,\sim} \arrow[l, "\sim"']
\end{tikzcd}
$$

where the categories in red are the replacement for “naturally occurring” categories in the middle.

The approach

- $\mathbf{Trb}_{\pi,\sim}$: tribes equivalent to π-tribes, and morphisms that becomes morphisms of lcc quasicategories upon applyin $\mathbf{Ho}_\infty$.

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- $\mathbf{scTrb}_{\pi,\sim}$ semi-cubical counterpart of $\mathbf{Trb}_{\pi,\sim}$.
- $\mathbf{scTrb}^{p}_{\pi}$: full subcategory of $\mathbf{scTrb}_{\pi}$ spanned by the tribe $\mathcal{T}$ such that $\iota_{\mathcal{T}} : \mathcal{T} \to P\mathcal{T}$ is π-closed.

The rigidification tool

Lemma

Consider a morphism $f : \mathcal{T} \to \mathcal{S}$ between π-tribes in $\mathbf{scTrb}_\pi^\sim$. Then, there exists a diagram

$$\begin{tikzcd} & \mathcal{T} \arrow[dl, "id_{\mathcal{T}}"'] \arrow[dd, "m"] \arrow[dr, "f"] & \\ \mathcal{T} & & \mathcal{S} \\ & \mathcal{T}' \arrow[ul, "p_0"] \arrow[ur, "p_1"'] & \end{tikzcd}$$

where $\mathcal{T}'$ is a π-tribe equivalent to $\mathcal{T}$ and the morphisms $\mathcal{T}' \to \mathcal{T}$ and $\mathcal{T}' \to \mathcal{S}$ are π-closed.

The rigidification tool

Proof.

Form the following pullback square:

$$\begin{tikzcd} \mathcal{T}' \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P\mathcal{S} \arrow[d] \\ \mathcal{T} \times \mathcal{S} \arrow[r, "f \times id_{\mathcal{S}}"'] & \mathcal{S} \times \mathcal{S} \end{tikzcd}$$

$\square$

$\mathbf{Trb}_{\pi,\sim} \to \mathbf{QCat}_{lcc}$

- On hom-spaces, we form the following pullback:

$$\begin{tikzcd} Hom_{\mathbf{Trb}_{\pi,\sim}}(X,Y) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & Hom_{\mathbf{QCat}_{lcc}}(\mathbf{Ho}_\infty(X), \mathbf{Ho}_\infty(Y)) \arrow[d, two heads] \\ Hom_{\mathbf{Trb}}(X,Y) \arrow[r, "\sim"] & Hom_{\mathbf{QCat}_{lex}}(\mathbf{Ho}_\infty(X), \mathbf{Ho}_\infty(Y)) \end{tikzcd}$$

- On object, this is not difficult.

$\mathbf{scTrb} \to \mathbf{Trb}$

Definition

For $\mathcal{T}$ a (π-)tribe, we define the category of semi-cubical frames

$$\mathrm{scFr}\mathcal{T} := \mathcal{T}_R^{\square^{op}}{}_{\sharp}$$

as the category of Reedy fibrant homotopical semi-cubical diagrams in $\mathcal{T}$

scTrb $\to$ Trb

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Proposition

$scFr\mathcal{T}$ is a semi-cubical (π-)tribe and the evaluation functor

$$\mathbf{ev}_0 : scFr\mathcal{T} \to \mathcal{T}$$

is a DK-equivalence.

$\mathbf{scTrb}_\pi^p \to \mathbf{scTrb}_\pi$

Remark

From the type theoretic point-of-view, the functor

$$\mathrm{scFr} : \mathbf{Trb}_\pi \to \mathbf{scTrb}_\pi$$

is connected to the free parametric[7] model associated to a given model of MLTT.

Proposition

For $\mathcal{T}$ a π-tribe, scFr$\mathcal{T}$ lies in $\mathbf{scTrb}_\pi^p$.

[7] Hugo Moeneclaey. "Parametricity and semi-cubical types". In: *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE. 2021, pp. 1–11.

$\mathbf{scTrb}^p_\pi \to \mathbf{scTrb}_{\pi,\sim}$

Lemma

Let $\mathcal{T}$ and $\mathcal{S}$ be π-tribes. A morphism $f : \mathcal{T} \to \mathcal{S}$ in $\mathbf{Ho}(\mathbf{scTrb}_{\pi,\sim})$ can be represented by a spans

$$\begin{tikzcd} & \mathcal{T}' \arrow[dl, two heads, "f_0"'] \arrow[dr, "f_1"] & \\ \mathcal{T} & & \mathcal{S} \end{tikzcd}$$

where $\mathcal{T}'$ is a π-tribe. Here, f_0 is weak equivalence in $\mathbf{scTrb}_{\pi,\sim}$, and f_1 is a map in $\mathbf{scTrb}_{\pi,\sim}$.

$\mathbf{scTrb}^{p}_{\pi} \to \mathbf{scTrb}_{\pi,\sim}$

Lemma

$\mathbf{Ho}(\mathbf{scTrb}_{\pi}) \to \mathbf{Ho}(\mathbf{scTrb}_{\pi,\sim})$ *is full.*

Proof.

Using the rigidification tool, we have

$$\begin{tikzcd}
& \mathcal{T}' \arrow[dl, "id_{\mathcal{T}'}"'] \arrow[dr, "f_1"] \arrow[dd, "\sim"] & \\
\mathcal{T}' & & \mathcal{S} \\
& \mathcal{T}'' \arrow[ul, "p_{1,0}"] \arrow[ur, "p_{1,1}"'] &
\end{tikzcd}$$

Thank you for you attention!

References

[1] Clark Barwick and Daniel M Kan. "Relative categories: another model for the homotopy theory of homotopy theories". In: *Indagationes Mathematicae* 23.1-2 (2012), pp. 42–68.

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[7] Peter LeFanu Lumsdaine and Michael A Warren. "The local universes model: an overlooked coherence construction for dependent type theories". In: *ACM Transactions on Computational Logic (TOCL)* 16.3 (2015), pp. 1–31.

[8] Hugo Moeneclaey. "Parametricity and semi-cubical types". In: *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE. 2021, pp. 1–11.