

Internal languages of locally cartesian closed $(\infty, 1)$ -categories

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Outline

- 1 The internal language conjectures
- 2 Some context for the problem
 - Tribes and clans
 - Strictification and rigidification
- 3 Establishing the conjecture
 - Working with fibration categories
 - The approach
 - Proofs

From relative categories to quasicategories

Relative categories

RelCat

and quasicategories

QCat

are both objects that combine homotopy and category. They are in fact equivalent framework to talk about $(\infty, 1)$ -categories.

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Definition

The $\mathbf{Ho}_\infty$ functor is defined as the following composite:

RelCat

↓ simplicial localization

Cat_Δ

↓ fibrant replacement

Cat_Δ

↓ $\mathbf{N}_\Delta$ (simplicial/homotopy-coherent nerve)

QCat

From relative categories to quasicategories

Theorem (Barwick-Kan^[1])

The functor

$$\mathbf{Ho}_\infty : \mathbf{RelCat} \rightarrow \mathbf{QCat}$$

is a DK-equivalence.

[1] Clark Barwick and Daniel M Kan. “Relative categories: another model for the homotopy theory of homotopy theories”. In: *Indagationes Mathematicae* 23.1-2 (2012), pp. 42–68, Clark Barwick and Daniël M Kan. “A Thomason-like Quillen equivalence between quasi-categories and relative categories”. In: *arXiv preprint arXiv:1101.0772* (2011).

Other $(\infty, 1)$ -categories frameworks

- Model categories
- Fibration/cofibration categories
- **Kan**-enriched categories
- Complete Segal Spaces

The conjectures

Conjecture (Internal language conjectures^[2])

The functor $\mathbf{Ho}_\infty$ restricts to DK-equivalences

$$\mathbf{CompCat}_{\Sigma, Id} \rightarrow \mathbf{Qcat}_{lex}^{[3]}$$

$$\mathbf{CompCat}_{\Sigma, \Pi_{ext}, Id} \rightarrow \mathbf{Qcat}_{lcc}$$

[2] Krzysztof Kapulkin and Peter LeFanu Lumsdaine. “The homotopy theory of type theories”. In: *Advances in Mathematics* 337 (2018), pp. 1–38.

[3] Krzysztof Kapulkin and Karol Szumiło. “Internal languages of finitely complete $(\infty, 1)$ -categories”. In: *Selecta Mathematica* 25.2 (2019), pp. 1–46

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where the domain (resp. codomain) categories have morphisms that preserves the structure involved up to isomorphism (resp. equivalence).

[2] Krzysztof Kapulkin and Peter LeFanu Lumsdaine. “The homotopy theory of type theories”. In: *Advances in Mathematics* 337 (2018), pp. 1–38.

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Clans

The notion of clan^[4] essentially axiomatizes/rephrases the structure on a category $\mathcal{C}$ equipped with a full comprehension structure, which is needed to interpret the core of type theory.

[4] [André Joyal](#). “Notes on clans and tribes”. In: *arXiv preprint arXiv:1710.10238* (2017).

Clans

The notion of clan^[4] essentially axiomatizes/rephrases the structure on a category $\mathcal{C}$ equipped with a full comprehension structure, which is needed to interpret the core of type theory.

Definition

A clan structure on a category $\mathcal{C}$ with a terminal object $\mathbf{1}$ is given by a class of maps $\mathcal{F}$ called fibrations such that:

- Isomorphisms are fibrations, $X \rightarrow \mathbf{1}$ is a fibration for every X .
- Fibrations are closed under composition. Pullbacks of fibrations exist and yield fibrations.

[4] [André Joyal](#). “Notes on clans and tribes”. In: *arXiv preprint arXiv:1710.10238* (2017).

Tribes

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- Every map factors as an anodyne map followed by a fibration.
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We will also consider π -tribes, which are essentially tribes such that every fibration admits an *internal product* along any fibration.

Essentially, this means that the underlying type theory also has Π -types.

Canonical comprehension

Given a π -tribe $\mathcal{T}$, the canonical comprehension structure given by the Grothendieck fibration

$$\mathbf{cod} : \mathcal{T}_{\text{fib}}^{\rightarrow} \rightarrow \mathcal{T}$$

supports Σ - and Π -types that are stable under pullback up to isomorphism.

Substitution is well-defined and functorial up to isomorphism.

Strictification

A **strictification**^[5] procedure aims at replacing this comprehension category by an equivalent split one, and such that Σ - and Π -types are strictly stable under substitution.

Pictorially:

Isomorphisms $\Longrightarrow$ Equalities

[5] Peter LeFanu Lumsdaine and Michael A Warren. “The local universes model: an overlooked coherence construction for dependent type theories”. In: *ACM Transactions on Computational Logic (TOCL)* 16.3 (2015), pp. 1–31.

Coherence in an ∞ -category

Consider a locally cartesian closed $(\infty,1)$ -category $\mathcal{C}$ (e.g. a quasicategory) with finite limits.

- Pullbacks give a substitution operation which is well-defined and functorial up to (homotopy) equivalence.
- Dependent products are also defined (and pullback-stable) up to equivalence.

Rigidification

A **rigidification** procedure aims at replacing $\mathcal{C}$ by a π -tribe presenting the same $(\infty,1)$ -category (up to equivalence).

Pictorially:

Equivalences $\Longrightarrow$ Isomorphisms

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DK-equivalences from equivalence of categories

Theorem (Cisinski^[6])

Given fibration categories $\mathcal{F}_0$ and $\mathcal{F}_1$, as well as an exact functor $H : \mathcal{F}_0 \rightarrow \mathcal{F}_1$, the following are equivalent:

- H is a DK-equivalence.
- $\mathbf{Ho}(H) : \mathbf{Ho}(\mathcal{F}_0) \rightarrow \mathbf{Ho}(\mathcal{F}_1)$ is an equivalence of categories.

[6] Denis-Charles Cisinski. “Catégories dérivables”. In: *Bulletin de la société mathématique de France* 138.3 (2010), pp. 317–393.

Fibration category of tribes

Trb can be equipped with a notion of fibration making it “almost” a fibration category in that, given a tribe $\mathcal{T}$, there is a canonical tribe $P\mathcal{T}$ whose objects are fibrant homotopical spans in $\mathcal{T}$.

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Trb can be equipped with a notion of fibration making it “almost” a fibration category in that, given a tribe $\mathcal{T}$, there is a canonical tribe $P\mathcal{T}$ whose objects are fibrant homotopical spans in $\mathcal{T}$. For every object x , we may choose a path object px (in $\mathcal{T}$) for x :

$$\begin{array}{c}
 & & & & x \\
 & & & \nearrow \sim & \\
 x & \overset{\sim}{\dashrightarrow} & px & \nearrow \sim & \\
 & & \searrow \sim & & x
 \end{array}$$

Fibration category of tribes

We then have a mapping $i : \mathcal{T} \rightarrow P\mathcal{T}$ fitting in a commutative triangle

$$\begin{array}{ccc} & P\mathcal{T} & \\ \begin{array}{c} \nearrow i \\ \text{red arrow} \end{array} & & \searrow \langle \pi_1, \pi_2 \rangle \\ \mathcal{T} & \xrightarrow{\Delta} & \mathcal{T} \times \mathcal{T} \end{array}$$

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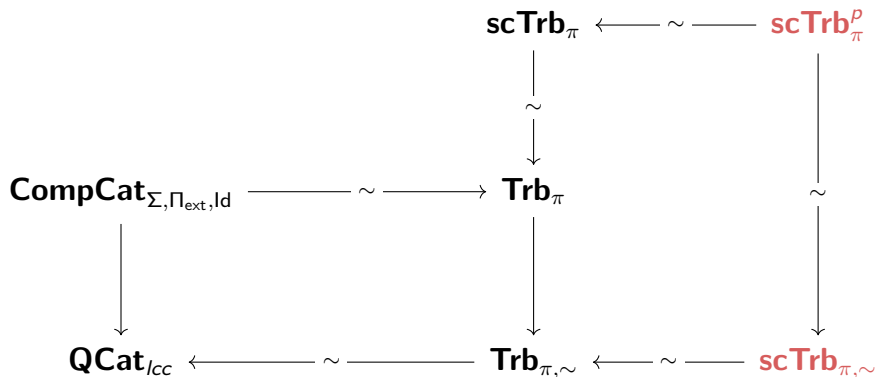
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However, the choice made need not imply that the mapping i is functorial. This is how we fall short of constructing a path object for $\mathcal{T}$, the only thing left needed for **Trb** to be a fibration category.

If $\mathcal{T}$ is a semi-cubical tribe, taking $Px := x^{\square^1}$ makes it functorial, and we write $\iota_{\mathcal{T}} : \mathcal{T} \rightarrow P\mathcal{T}$

The approach

To prove the second part of the conjecture, our approach is the following:



where the categories in red are the replacement for “naturally occurring” categories in the middle.

The approach

- **Trb** $_{\pi, \sim}$: tribes equivalent to π -tribes, and morphisms that becomes morphisms of lcc quasicategories upon applyin **Ho** $_{\infty}$.

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- **scTrb** $_{\pi}$: semi-cubical π -tribes.
- **scTrb** $_{\pi, \sim}$ semi-cubical counterpart of **Trb** $_{\pi, \sim}$.

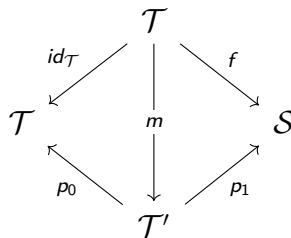
The approach

- $\mathbf{Trb}_{\pi, \sim}$: tribes equivalent to π -tribes, and morphisms that becomes morphisms of lcc quasicategories upon applyin $\mathbf{Ho}_{\infty}$.
- $\mathbf{scTrb}_{\pi}$: semi-cubical π -tribes.
- $\mathbf{scTrb}_{\pi, \sim}$ semi-cubical counterpart of $\mathbf{Trb}_{\pi, \sim}$.
- $\mathbf{scTrb}_{\pi}^P$: full subcategory of $\mathbf{scTrb}_{\pi}$ spanned by the tribe $\mathcal{T}$ such that $\iota_{\mathcal{T}} : \mathcal{T} \rightarrow P\mathcal{T}$ is π -closed.

The rigidification tool

Lemma

Consider a morphism $f : \mathcal{T} \rightarrow \mathcal{S}$ between π -tribes in $\mathbf{scTrb}_{\pi}^{\sim}$. Then, there exists a diagram



where $\mathcal{T}'$ is a π -tribe equivalent to $\mathcal{T}$ and the morphisms $\mathcal{T}' \rightarrow \mathcal{T}$ and $\mathcal{T}' \rightarrow \mathcal{S}$ are π -closed.

The rigidification tool

Proof.

Form the following pullback square:

$$\begin{array}{ccc} \mathcal{T}' & \xrightarrow{u} & PS \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{T} \times \mathcal{S} & \xrightarrow{f \times id_{\mathcal{S}}} & \mathcal{S} \times \mathcal{S} \end{array}$$



$\mathbf{Trb}_{\pi, \sim} \rightarrow \mathbf{QCat}_{lcc}$

- On hom-spaces, we form the following pullback:

$$\begin{array}{ccc}
 \mathrm{Hom}_{\mathbf{Trb}_{\pi, \sim}}(X, Y) & \longrightarrow & \mathrm{Hom}_{\mathbf{QCat}_{lcc}}(\mathbf{Ho}_{\infty}(X), \mathbf{Ho}_{\infty}(Y)) \\
 \downarrow & \lrcorner & \downarrow \\
 \mathrm{Hom}_{\mathbf{Trb}}(X, Y) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbf{QCat}_{lex}}(\mathbf{Ho}_{\infty}(X), \mathbf{Ho}_{\infty}(Y))
 \end{array}$$

- On object, this is not difficult.

scTrb $\rightarrow$ Trb

Definition

For $\mathcal{T}$ a (π) -tribe, we define the category of semi-cubical frames

$$\text{scFr}\mathcal{T} := \mathcal{T}_R^{\square\#op}$$

as the category of Reedy fibrant homotopical semi-cubical diagrams in $\mathcal{T}$

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Proposition

scFr $\mathcal{T}$ is a semi-cubical (π) -tribe and the evaluation functor

$$\mathbf{ev}_0 : \text{scFr}\mathcal{T} \rightarrow \mathcal{T}$$

is a DK-equivalence.

$$\mathbf{scTrb}_{\pi}^p \rightarrow \mathbf{scTrb}_{\pi}$$

Remark

From the type theoretic point-of-view, the functor

$$\mathbf{scFr} : \mathbf{Trb}_{\pi} \rightarrow \mathbf{scTrb}_{\pi}$$

is connected to the free parametric^[7] model associated to a given model of MLTT.

Proposition

For $\mathcal{T}$ a π -tribe, $\mathbf{scFr}\mathcal{T}$ lies in $\mathbf{scTrb}_{\pi}^p$.

[7] Hugo Moeneclaey. “Parametricity and semi-cubical types”. In: *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE, 2021, pp. 1–11.

$$\mathbf{scTrb}_{\pi}^p \rightarrow \mathbf{scTrb}_{\pi, \sim}$$

Lemma

Let $\mathcal{T}$ and $\mathcal{S}$ be π -tribes. A morphism $f : \mathcal{T} \rightarrow \mathcal{S}$ in $\mathbf{Ho}(\mathbf{scTrb}_{\pi, \sim})$ can be represented by a spans

$$\begin{array}{ccc} & \mathcal{T}' & \\ f_0 \swarrow & & \searrow f_1 \\ \mathcal{T} & & \mathcal{S} \end{array}$$

where $\mathcal{T}'$ is a π -tribe. Here, f_0 is weak equivalence in $\mathbf{scTrb}_{\pi, \sim}$, and f_1 is a map in $\mathbf{scTrb}_{\pi, \sim}$.

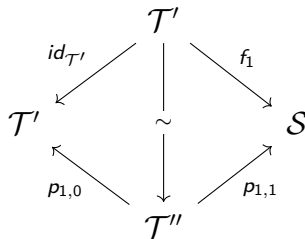
$$\mathbf{scTrb}_{\pi}^p \rightarrow \mathbf{scTrb}_{\pi, \sim}$$

Lemma

$\mathbf{Ho}(\mathbf{scTrb}_{\pi}) \rightarrow \mathbf{Ho}(\mathbf{scTrb}_{\pi, \sim})$ is full.

Proof.

Using the rigidification tool, we have



Thank you for you attention!

References

- [1] Clark Barwick and Daniel M Kan. “Relative categories: another model for the homotopy theory of homotopy theories”. In: *Indagationes Mathematicae* 23.1-2 (2012), pp. 42–68.
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