Constructive ordinals

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Section 1

Introduction

Motivation

The notion of a well-order is one of the fundamental notions of logic. An ordinal is essentially a canonical representative of an isomorphism class of well-orders. So we expect it to be very important both classically and constructively.

Ordinals are also heavily involved in proving some important theorems.

Theorem (Adámek)

Let F be an endofunctor on a cocomplete category $\mathcal{C}$, and α be a limit ordinal. If F preserves colimits of shape α, then F has an initial algebra.

Theorem (Quillen, Bourke & Garner)

Let $\mathcal{C}$ be a locally small and cocomplete category, and α be a limit ordinal. If $\mathbb{A}$ is a small double category over $\mathcal{C}$ such that Aa is α-presentable for each object a in $\mathbb{A}$, then the algebraic weak factorization system cofibrantly generated by $\mathbb{A}$ exists.

Ordinals

As it turns out, a pretty good constructive notion of ordinal already exists. It can be found in:

- constructive set theory **CZF**, as the notion of a transitive set of transitive sets (due to Powell, I believe).
- homotopy type theory, see the Book.
- Joyal and Moerdijk's book on algebraic set theory (what are called the "von Neumann ordinals" there).

One of points of this presentation is to say that constructive versions of the theorems on the previous slide can be proved on the basis of this constructive notion of ordinals.

Warning 1: These are different from Paul Taylor's plumb ordinals.
Warning 2: The classical mathematician should be aware that the following principles are TABOO:

- Linearity: for all ordinals α, β either $\alpha \leq \beta$ or $\beta \leq \alpha$.
- Least Ordinal Principle: every inhabited class of ordinals has a least element.

Section 2

Constructive ordinals

A synthetic account of ordinals

We assume the ordinals Ord come equipped with operations:

$$\begin{aligned} s &: \mathrm{Ord} \to \mathrm{Ord} \\ \bigcup &: \mathrm{Pow}(\mathrm{Ord}) \to \mathrm{Ord} \end{aligned}$$

and an operation

$$\Downarrow \;:\; \mathrm{Ord} \to \mathrm{Pow}(\mathrm{Ord})$$

satisfying:

$$\begin{aligned} \Downarrow(s\alpha) &:= \{\alpha\} \cup \Downarrow\alpha, \\ \Downarrow(\bigcup A) &:= \bigcup_{\alpha \in A} \Downarrow\alpha. \end{aligned}$$

In addition, we assume the following inference rules:

$$(\mathrm{Induction}) \quad \frac{(\forall\alpha)\left(\varphi(\alpha) \to \varphi(s\alpha)\right) \qquad (\forall A)\left((\forall\alpha \in A)\,\varphi(\alpha) \to \varphi(\bigcup A)\right)}{(\forall\alpha)\,\varphi(\alpha)} \qquad (\mathrm{Extensionality}) \quad \frac{\Downarrow\alpha = \Downarrow\beta}{\alpha = \beta}$$

A strict ordering

We write

$$\beta < \alpha \quad :\equiv \quad \beta \in \Downarrow \alpha.$$

Then we have:

$$\beta < s\alpha \quad \Leftrightarrow \quad \beta = \alpha \vee \beta < \alpha.$$
$$\beta < \bigcup A \quad \Leftrightarrow \quad (\exists \alpha \in A)\, \beta < \alpha.$$

We can establish the following properties of $<$ by induction.

(1) $\Downarrow \alpha$ is a set for each ordinal α.

(2) $<$ is transitive. (Hint: show $\gamma < \beta < \alpha \Rightarrow \gamma < \alpha$ by induction α.)

(3) The following $<$-induction scheme is valid:

$$\frac{(\forall \alpha)\,\big(\,(\forall \beta < \alpha)\,\psi(\beta) \to \psi(\alpha)\,\big)}{(\forall \alpha)\,\psi(\alpha)}$$

(Hint: prove $(\forall \beta < \alpha)\,\psi(\beta)$ by induction on α.)

The inclusion ordering

We can also define the following inclusion ordering:

$$\alpha \subseteq \beta :\equiv (\forall \gamma) \left(\gamma < \alpha \Rightarrow \gamma < \beta \right).$$

Lemma

The relation $\subseteq$ partially orders Ord, and $\bigcup$ computes small suprema with respect to this ordering.

Proposition

$(\mathrm{Ord}, \subseteq, s)$ is the initial ZF-algebra with a progressive successor (that is, with $\alpha \subseteq s\alpha$ holding for all α).

Note that the two orderings are related as follows:

$$\alpha < \beta \iff s\alpha \subseteq \beta.$$

A third ordering

Finally, we define $\alpha \leq \beta :\equiv \alpha < \beta \vee \alpha = \beta$.

Lemma

The relation $\leq$ is a partial order.

The proof of this relies on:

Lemma

There are no ordinals α, β such that $\alpha \subseteq \beta < \alpha$ holds. In particular, $\alpha < \alpha$ never holds.

Proof.

Show $\neg(\exists\beta)\,(\,\alpha \subseteq \beta < \alpha\,)$ by induction on α. $\square$

Remark: $\alpha \leq \beta$ implies $\alpha \subseteq \beta$, but the converse is TABOO.
Remark: In what follows I will always regard $\Downarrow \alpha$ as a poset (category) ordered by $\leq$.

Section 3

Adámek's fixed point theorem

Adámek's fixed point theorem

In what follows I will give a constructive account of:

Theorem (Adámek)

Let F be an endofunctor on a cocomplete category $\mathcal{C}$, and α be a limit ordinal. If F preserves colimits of shape α, then F has an initial algebra.

With a slightly different presentation, this proof can be found in the work of Pitts & Steenkamp. Indeed:

- I will use the language of algebraic chains (wth the caveat that constructively they are not chains).
- I have changed "inflationary iteration" to "progressive iteration".

Algebraic chain

Suppose $F : \mathcal{C} \to \mathcal{C}$ is an endofunctor on a category $\mathcal{C}$ with a choice of small colimits.

We define an *algebraic chain* of length α to be a pair (X, x) consisting of:

(a) a functor $X : \Downarrow \alpha \to \mathcal{C}$,

(b) for any $\gamma < \beta < \alpha$ a map $x_\gamma^\beta : FX_\gamma \to X_\beta$ in $\mathcal{C}$,

such that for all $\delta < \gamma < \beta$ the two diagrams on the left commute:

$$\begin{tikzcd} FX_\gamma \arrow[r, "x_\gamma^\beta"] & X_\beta \\ FX_\delta \arrow[u, "FX_\delta^\gamma"] \arrow[ur, "x_\delta^\beta"'] & \end{tikzcd} \qquad \begin{tikzcd} & X_\beta \\ FX_\delta \arrow[ur, "x_\delta^\beta"] \arrow[r, "x_\delta^\gamma"'] & X_\gamma \arrow[u, "X_\gamma^\beta"'] \end{tikzcd} \qquad \begin{tikzcd} FX_\gamma \arrow[r, "F\varphi_\gamma"] \arrow[d, "x_\gamma^\beta"] & FY_\gamma \arrow[d, "y_\gamma^\beta"] \\ X_\beta \arrow[r, "\varphi_\beta"] & Y_\beta \end{tikzcd}$$

A morphism of chains $\varphi : (X, x) \to (Y, y)$ is a natural transformation $\varphi : X \to Y$ making the diagram on the right commute for any $\gamma < \beta < \alpha$.

Progressive iteration

Lemma

Let (X, x) be an algebraic chain of length α. If $\beta \leq \alpha$, then there is a diagram $\Downarrow \beta \to \mathcal{C}$ obtained by sending γ to FX_γ and $\delta < \gamma$ to FX_δ^γ. If $\beta < \alpha$, then we have a cocone on this diagram given by $(X_\beta, x_\gamma^\beta : FX_\gamma \to X_\beta)$.

Definition

An algebraic chain (X, x) will be called the *progressive iteration of F over α* if for each $\beta < \alpha$ the cocone from the lemma above is the chosen colimit.

Then one proves:

1. The progressive iteration of F over α is initial in the category of algebraic chains of length α.
2. For each ordinal α there is exists a unique progessive iteration of F over α.

Constructive initial algebra theorem

Definition

An ordinal κ will be called a *size* if for each $\alpha, \beta < \kappa$ there exists a $\gamma < \kappa$ such that $\alpha < \gamma$ and $\beta < \gamma$.

Theorem

Let $\mathcal{C}$ be a cocomplete category and κ be a size. If $F : \mathcal{C} \to \mathcal{C}$ is an endofunctor preserving colimits of shape κ, then the initial F-algebra exists.

Proof.

Let (X, x) be the unique progressive iteration of F over κ. Then $I = \operatorname{colim} X$ can be equipped with structure of an F-algebra which will make it initial in the category of F-algebras. □

Section 4

A constructive small object argument

Variations

By modifying the notion of progressive iteration we can show the following variations.

Theorem

Let $\mathcal{C}$ be a cocomplete category and κ be a size. If $F : \mathcal{C} \to \mathcal{C}$ is a pointed endofunctor preserving colimits of shape κ, then the initial pointed F-algebra exists.

Theorem

Let $\mathcal{C}$ be a cocomplete category and κ be a size. If $F : \mathcal{C} \to \mathcal{C}$ is a special endofunctor preserving colimits of shape κ, then the initial special F-algebra exists.

Here we assume we have natural transformations $\eta : 1 \to F$ and $\lambda : G \to FF$ and $\mu : G \to F$ and the following commutative diagrams for algebras $(X, x : FX \to X)$:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "1"'] & FX \arrow[d, "x"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} GX \arrow[r, "\lambda_X"] \arrow[d, "\mu_X"] & FFX \arrow[r, "Fx"] & FX \arrow[d, "x"] \\ FX \arrow[rr, "x"] & & X \end{tikzcd}$$

Constructive small object argument

The latter can be used to show the following theorem (jww Paul Seip and John Bourke).

Theorem (Constructive small object argument)

Let $\mathcal{C}$ be a locally small and cocomplete category, and κ be a size. If $\mathbb{A}$ is a small double category over $\mathcal{C}$ such that Aa is κ-presentable for each object a in $\mathbb{A}$, then the algebraic weak factorisation system cofibrantly generated by $\mathbb{A}$ exists.

Here we have used the following definition:

Definition

Let $\mathcal{C}$ be a category and κ be a size. We say that an object A in $\mathcal{C}$ is κ-*presentable*) if

$$\mathrm{Hom}(A, -) : \mathcal{C} \to \mathsf{Sets}$$

preserves κ-filtered colimits.

The need for good WISC-y axioms

However, we need a bit of choice to apply this result.

Theorem

In classical Zermelo-Fraenkel set theory **ZF** (without choice), it cannot be proved that there is a cardinal κ such that $\mathbb{N}$ is κ-presentable in the category of sets. Indeed, there is a small double category over sets in which Aa is always countable and which cannot be proved to generate an algebraic weak factorisation system in **ZF**.

Conjecture

Using **WISC**, we can show that every set is κ-presentable for some size κ, so then we should be able to avoid these issues.

Section 5

Conclusion

Conclusion

- A satisfying theory of ordinals supporting useful transfinite arguments can be developed.
- In the future it might be good to see if we can constructivise Kelly and write a paper on "A unified constructive treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on"
- And the same applies to the literature on κ-presentable and κ-accessible categories!

Thank you for your attention!